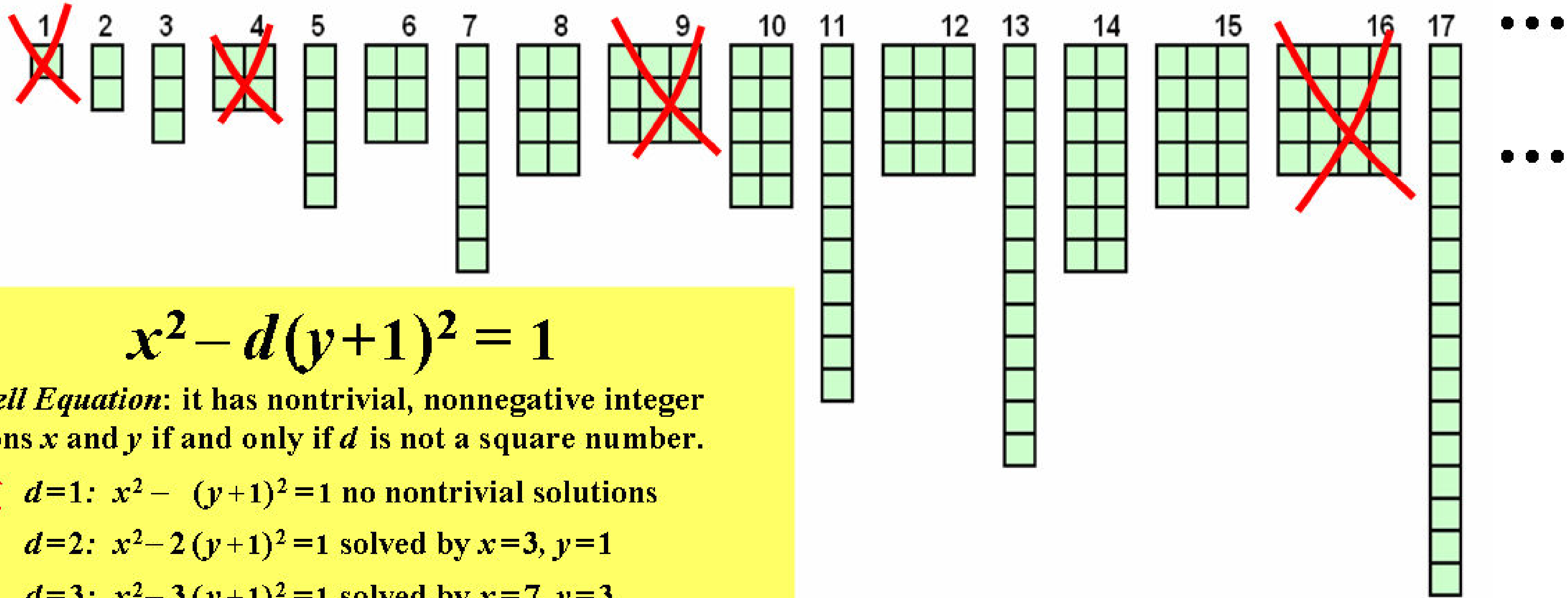
THEOREM OF THE DAY

The DPRM Theorem *Every recursively enumerable set is Diophantine.*



$$x^2 - d(y+1)^2 = 1$$

The *Pell Equation*: it has nontrivial, nonnegative integer solutions x and y if and only if d is not a square number.

$d=1$: $x^2 - (y+1)^2 = 1$ no nontrivial solutions

$d=2$: $x^2 - 2(y+1)^2 = 1$ solved by $x=3, y=1$

$d=3$: $x^2 - 3(y+1)^2 = 1$ solved by $x=7, y=3$

$d=4$: $x^2 - 4(y+1)^2 = 1$ no nontrivial solutions

$d=5$: $x^2 - 5(y+1)^2 = 1$ solved by $x=9, y=3$

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A set is *recursively enumerable* if there exists some (possibly non-terminating) algorithm which lists precisely the members of the set. For instance, the non-square positive integers can be so enumerated, merely by checking every integer in turn. A set S of integers is *Diophantine* if it can be specified by a polynomial equation $E(x_i)$, in variables $d, x_1, \ldots, x_t, t \geq 1$, and with integer coefficients, such that E can be solved by giving integer values to $x_1, \ldots, x_t$ if and only if d is assigned a value in S. More generally, S may consist of n-tuples, $(d_1, \ldots, d_n)$. The famous Pell equation, for example, shows that the set of non-square integers is Diophantine. It is a classic result that not all recursively enumerable sets have a finitely terminating algorithm to test membership. The DPRM Theorem therefore implies that some Diophantine sets have no membership test and therefore that there exist Diophantine equations whose solvability cannot be checked algorithmically in a finite number of steps.

The DPRM Theorem is named for Martin Davis, Hilary Putnam, and Julia Robinson who spent a lifetime developing the machinery necessary to assert the theorem, and Yuri Matiyasevich who, as a PhD student in 1971, made ingenious use of the Fibonacci sequence to deliver the coup de grace to the tenth of Hilbert's famous millenium problems.

Web link: Bjorn Poonen's 2011 Chauvenet Prize winning paper at www.maa.org/programs/maa-awards/writing-awards/chauvenet-prizes.

Further reading: *Hilbert's Tenth Problem* by Yuri Matiyasevich, MIT Press, 1993.