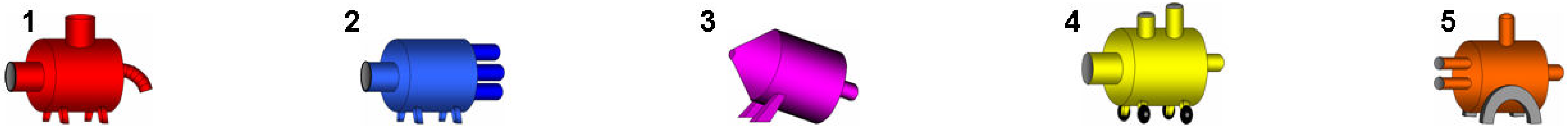
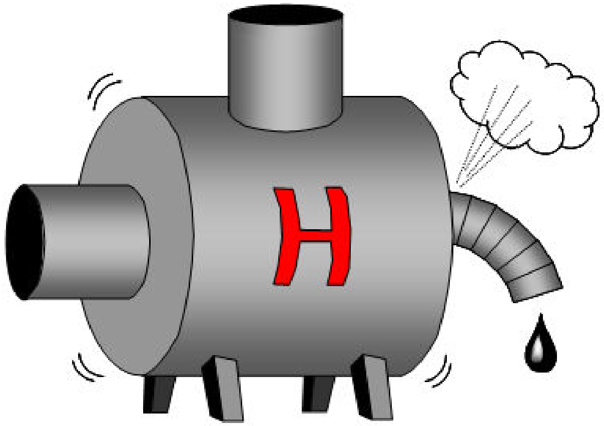
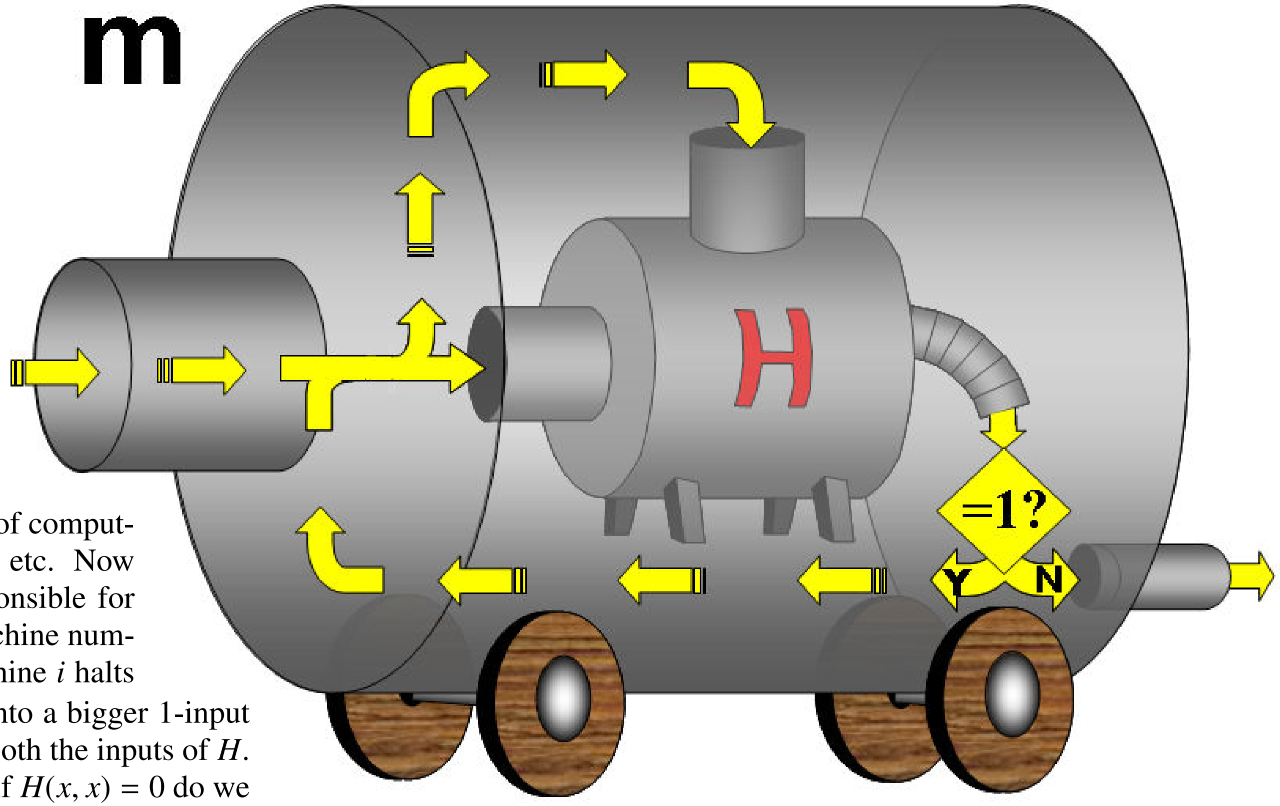
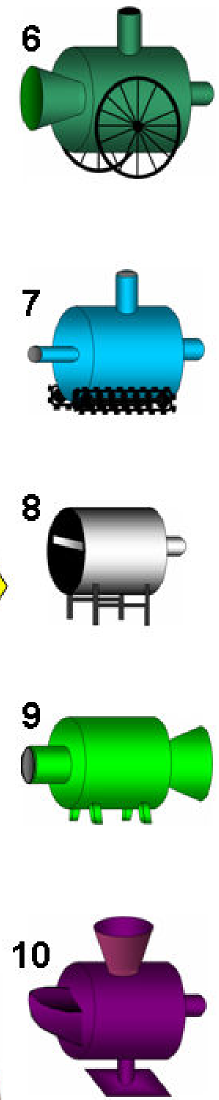
THEOREM OF THE DAY



The Insolvability of the Entscheidungsproblem *No adequate axiomatisation $\mathcal{S}$ of mathematics can provide an algorithm which tests statements in $\mathcal{S}$ for provability from the axioms of $\mathcal{S}$.*

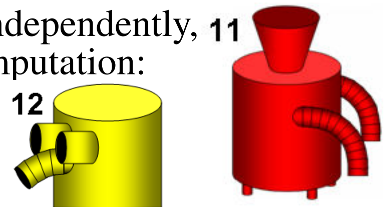






Let us say $\mathcal{S}$ is adequate if it captures arithmetic. We must also say what we mean by an ‘algorithm’: an accepted definition is an exhaustive enumeration of computing ‘machines’: machine 1, machine 2, machine 3, etc. Now suppose one of the machines, we call it H, is responsible for declaring True or False the following statement: machine number i halts on input j. So we say $H(i, j) = 1$ if machine i halts on input j and $H(i, j) = 0$ otherwise. We build H into a bigger 1-input machine, called S, which takes its single input x as both the inputs of H. If $H(x, x)$ outputs 1 then we try input x again! Only if $H(x, x) = 0$ do we output a result. Machine S must appear in our enumeration, say it is machine number m. We can run S using m as input: what can we say about $S(m)$? By definition of H, we determine that $S(m)$ halts if and only if $H(m, m) = 1$ ... exactly the condition which makes machine S go into a loop! So $S(m)$ halts if and only if H says it does not halt, a contradiction which refutes the existence of H. Moreover, our adequate system $\mathcal{S}$ can be shown to supply a statement X_{ij} allowing “machine i halts on j” to be translated into “statement X_{ij} is provable in $\mathcal{S}$”. So, no $H \Rightarrow$ no Entscheidungsproblem solution.

The Entscheidungsproblem was posed by David Hilbert and Wilhelm Ackermann in 1928. In 1936, independently, Alonzo Church and Alan Turing gave negative solutions based on formal mathematical models of computation: the λ-calculus and the Turing machine, respectively. In the same year, Emil Post introduced the Post Machine, having already anticipated by twenty years many of the ideas of Church, Turing and Gödel.



Web link: www.karlin.mff.cuni.cz/~stovicek/math/decidability.pdf

Further reading: *The Essential Turing* by B. Jack Copeland (ed.), Oxford University Press, 2004.