THEOREM OF THE DAY

Goodstein's Theorem *For a positive integer M, derive the* hereditary base k *representation, in which every numeral is either k or zero, as follows:* (1) *write M in base k, as $M = a_0k^0+a_1k^1+\ldots+a_{n-1}k^{n-1}+a_nk^n$, for suitable a_i in $\{0,\ldots,k-1\}$;* (2) *for $0 \le i \le n$, write a_ik^i as a sum of a_i copies of k^i;* (3) *apply steps* (1) – (3) *to all occurrences of $1,\ldots,k-1$ in exponents. Now suppose the* Goodstein sequence *of M is obtained by applying the following algorithm, starting at base $k = 2$:*

> (G1) *write M in hereditary base k notation;*
> (G2) *replace every occurrence of 'k' by '$k+1$' and subtract 1 from the resulting number;*
> (G3) *if the result is zero then STOP; otherwise apply* (G1) – (G3) *to the result with base k replaced by base $k+1$.*

Then, for every positive integer M, the Goodstein sequence terminates.

Hereditary base 5 example: $15678 \xrightarrow{(1)} 3\times5^0+2\times5^2+1\times5^6 \xrightarrow{(2)} 5^0+5^0+5^0+5^2+5^2+5^6 \xrightarrow{(3)} 5^0+5^0+5^0+5^{5^0+5^0}+5^{5^0+5^0}+5^{5^0+5^1} \xrightarrow{(3)} 5^0+5^0+5^0+5^{5^0+5^0}+5^{5^0+5^0}+5^{5^0+5^{5^0}}$

value	3	3	3	2	1
k	2	3	4	5	6
hereditary base k	$2^{2^0}+2^0$	3^{3^0}	$4^0+4^0+4^0$	5^0+5^0	6^0
$k \to k+1$	$3^{3^0}+3^0$	4^{4^0}	$5^0+5^0+5^0$	6^0+6^0	7^0
subtract 1	3^{3^0}	$4^{4^0}-1$	5^0+5^0	6^0	0
result	3	3	2	1	0

value	6	29	257	3125	46
k	2	3	4	5	6
hereditary base k	$2^{2^{2^0}}+2^{2^0}$	$3^{3^{3^0}}+3^0+3^0$	$4^{4^{4^0}}+4^0$	$5^{5^{5^0}}$	…
$k \to k+1$	$3^{3^{3^0}}+3^{3^0}$	$4^{4^{4^0}}+4^0+4^0$	$5^{5^{5^0}}+5^0$	$6^{6^{6^0}}$	…
subtract 1	$3^{3^{3^0}}+3^{3^0}-1$	$4^{4^{4^0}}+4^0$	$5^{5^{5^0}}$	$6^{6^{6^0}}-1$	…
result	29	257	3125	46655	…

The Goodstein sequence for $M = 3$ is seen here to terminate after 5 iterations of the algorithm but this is the largest value of M for which the Goodstein sequence can feasibly be constructed: even for $M = 4$ it requires many more iterations than there are atoms in the universe!

Goodstein's 1944 theorem is important because it can be stated as a sentence in Peano arithmetic but no proof exists within this system. It is therefore a 'natural' example of Gödel's First Incompleteness Theorem in action.

Web link: old.nationalcurvebank.org/goodstein/goodstein.htm

Further reading: *An Introduction to Gdel's Theorems* by Peter Smith, Cambridge University Press, 2nd edition, 2013.