**THEOREM OF THE DAY**

**Goodstein’s Theorem** For a positive integer $M$, derive the hereditary base $k$ representation, in which every numeral is either $k$ or zero, as follows: (1) write $M$ in base $k$, as $M = a_0 k^0 + a_1 k^1 + \ldots + a_{n-1} k^{n-1} + a_n k^n$, for suitable $a_i$ in $\{0, \ldots, k - 1\}$; (2) for $0 \leq i \leq n$, write $a_i k^i$ as a sum of $a_i$ copies of $k^i$; (3) apply steps (1) – (3) to all occurrences of $1, \ldots, k - 1$ in exponents. Now suppose the Goodstein sequence of $M$ is obtained by applying the following algorithm, starting at base $k = 2$:

1. **(G1)** write $M$ in hereditary base $k$ notation;
2. **(G2)** replace every occurrence of ‘$k$’ by ‘$k + 1$’ and subtract 1 from the resulting number;
3. **(G3)** if the result is zero then STOP; otherwise apply (G1) – (G3) to the result with base $k$ replaced by base $k + 1$.

Then, for every positive integer $M$, the Goodstein sequence terminates.

**Hereditary base 5 example:**

1. $15678 \xrightarrow{(1)} 3 \times 5^0 + 2 \times 5^2 + 1 \times 5^6$
2. $3 \xrightarrow{(2)} 5^0 + 5^0 + 5^2 + 5^2 + 5^6$
3. $5 \xrightarrow{(3)} 5^0 + 5^0 + 5^0 + 5^0 + 5^0 + 5^0 + 5^0 + 5^0 + 5^0 + 5^0$

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>hereditary base $k$</td>
<td>$2^2+2^0$</td>
<td>$3^0$</td>
<td>$4^0+4^0+4^0$</td>
<td>$5^0+5^0$</td>
<td>$6^0$</td>
</tr>
<tr>
<td>$k \to k+1$</td>
<td>$3^3+3^0$</td>
<td>$4^0$</td>
<td>$5^0+5^0+5^0$</td>
<td>$6^0+6^0$</td>
<td>$7^0$</td>
</tr>
<tr>
<td>subtract 1</td>
<td>$3^3$</td>
<td>$4^0-1$</td>
<td>$5^0+5^0$</td>
<td>$6^0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The Goodstein sequence for $M = 3$ is seen here to terminate after 5 iterations of the algorithm but this is the largest value of $M$ for which the Goodstein sequence can feasibly be constructed: even for $M = 4$ it requires many more iterations than there are atoms in the universe!

Goodstein’s 1944 theorem is important because it can be stated as a sentence in Peano arithmetic but no proof exists within this system. It is therefore a ‘natural’ example of Gödel’s First Incompleteness Theorem in action.

**Web link:** old.nationalcurvebank.org/goodstein/goodstein.htm

**Further reading:** An Introduction to Formal Logic by Peter Smith, Cambridge University Press, 2003.