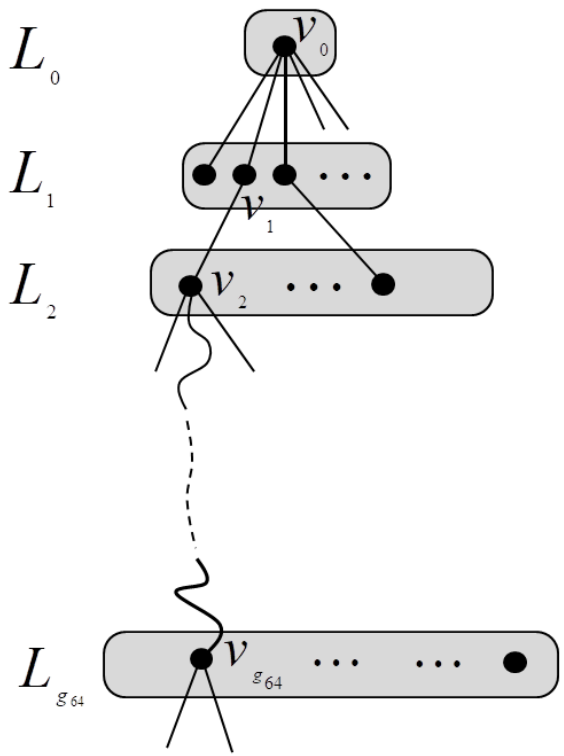
THEOREM OF THE DAY

Kőnig's Infinity Lemma *An infinite tree in which every level is finite contains an infinite path.*



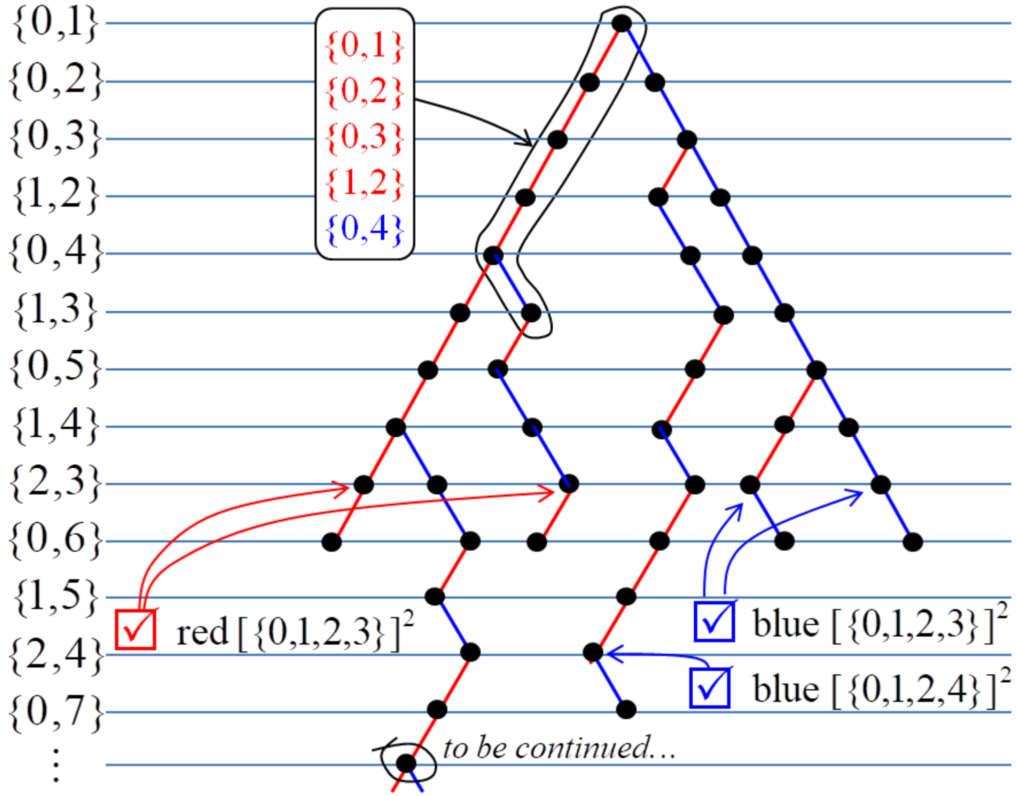
By **tree** we mean a collection of nodes arranged in **levels**, shown descending, in our drawings, from level L_0 which contains a single node, the **root**. For $i > 0$, each node in level L_i has an edge to a unique node in level L_{i-1}. There are no other edges.

The drawing on the right represents a tree with an infinite number of levels: a walk from $L_{g_{64}}$ up to the root v_0 will traverse g_{64} edges, where g_{64} is the unimaginably large **Graham's number**; and yet this is but an infinitesimal journey in the tree! Observe that, from however far down the tree, any node has just a single unique path up to the root, arriving via a unique node in level L_1. Suppose all tree levels are finite. An infinite number of nodes have paths up to the root passing through level L_1 which is finite. So by the Pigeonhole Principle, there is some node in L_1, say v_1, which is again the root of an infinite tree. Consider the path starting v_0, v_1. Applying the Pigeonhole Principle again we can add to this path a third infinite tree root, say v_2, in level L_2. Repeating this process indefinitely will confirm the existence of an infinite path, as asserted by Kőnig's lemma.

In 1926, Dénes Kőnig's lemma was a contribution to his search for a combinatorial proof of the Cantor–Schröder–Bernstein Theorem. Ten years later, when Kőnig published the first ever text book on graph theory, the lemma appeared in its own right as a basic combinatorial tool.

Web link: people.math.ethz.ch/~halorenz/4students/LogikML/Setting.pdf; Graham's number: research.phys.cmu.edu/biophysics/2021/01/09/.

Further reading: *Combinatorics and Graph Theory, 2nd edition* by John M. Harris, Jeffrey L. Hirst and Michael J. Mossinghoff Springer, 2008 (chapter 3)



We may apply Kőnig's lemma to the Infinite Ramsey Theorem, as applied to the set $[A]^2$ of unordered pairs of elements in a set A, to deduce the Finite Ramsey Theorem: *For any positive integer n there is a positive integer m such that, for any two-colouring of the elements of $[\{0,\ldots,m-1\}]^2$, there is a subset X of $\{0,\ldots,m-1\}$ of size n such that $[X]^2$ is monochromatic.*

Proof. We construct a tree whose levels correspond to some listing of the elements of $[\mathbb{N}]^2$. Descend from level L_0, the root. For each node x in level L_i add a red edge and a blue edge to distinct nodes in level L_{i+1}. If the path from the root through either of these edges now describes a monochromatic $[X]^2$ with $|X| = n$, mark x with a '✓' and add no further edges below x. Suppose Ramsey's Theorem is false, so there is an n for which no m guarantees the required set X of size n. This means at every level of our tree some node must have edges descending from it. So our tree is infinite. But its levels are finite so by Kőnig it has an infinite path. Now this path specifies a two-colouring of all of $[\mathbb{N}]^2$ having no monochromatic $[X]^2$ with $|X| = n$. But then there can certainly be no infinite subset X of $\mathbb{N}$ with monochromatic $[X]^2$, contradicting Infinite Ramsey.