THEOREM OF THE DAY

The Pigeonhole Principle *Let $q_1, q_2, \ldots, q_n$ be positive integers and let Q be a set of size $q_1 + \ldots + q_n - n + 1$. Take a mapping from Q to a set $P = \{p_1, \ldots, p_n\}$ of size n. Then, under this mapping,*

1. *for some i, $1 \le i \le n$, p_i is the image of at least q_i points in Q;*
2. *if any of the q_i are allowed to be infinity then some p_i in P is the image of infinitely many points in Q.*

0, 1	0, 2																		
0, 3	1, 2	0, 4	**1, 3**																
0, 5	**1, 4**	2, 3	0, 6	1, 5	2, 4														
0, 7	**1, 6**	2, 5	**3, 4**	0, 8	1, 7	2, 6	3, 5												
0, 9	1, 8	2, 7	**3, 6**	4, 5	0, 10	1, 9	2, 8	3, 7	**4, 6**										
0, 11	**1, 10**	2, 9	3, 8	4, 7	5, 6	0, 12	1, 11	2, 10	3, 9	4, 8	5, 7								
0, 13	1, 12	2, 11	**3, 10**	4, 9	5, 8	6, 7	0, 14	1, 13	2, 12	3, 11	**4, 10**	5, 9	6, 8						
0, 15	1, 14	2, 13	3, 12	4, 11	5, 10	6, 9	7, 8	0, 16	1, 15	2, 14	3, 13	4, 12	5, 11	6, 10	7, 9				
0, 17	1, 16	2, 15	3, 14	4, 13	5, 12	6, 11	7, 10	8, 9	0, 18	1, 17	2, 16	3, 15	4, 14	5, 13	6, 12	7, 11	8, 10		
0, 19	1, 18	2, 17	3, 16	4, 15	5, 14	6, 13	7, 12	8, 11	9, 10	0, 20	1, 19	2, 18	3, 17	4, 16	5, 15	6, 14	7, 13	8, 12	9, 11

Table of unordered pairs $\{i, m-i\}, 0 \le i \le \lfloor (m-1)/2 \rfloor$, with $m = 1 \ldots n$. The table is infinite ($n \to \infty$) but we have tabulated up to $n = 20$. Each table row covers two values of m, so the table has 10 rows. The pairs are partitioned as belonging in odd or even columns of the table. A challenge: find a large subset of numbers all of whose pairs lie in the even (yellow) columns. The boxed entries in the table perhaps look hopeful for the set $\{1, 3, 4, 6, 10\}$...

In fact, a set of five numbers can never have all pairs lying in even columns of the table! Symmetry might suggest a similar limit would hold for the odd columns as well but this is not true. We invoke an easy consequence of the infinite version of the Pigeonhole Principle, viz **The Infinite Ramsey Theorem** *Suppose that each of the two-element subsets of $\mathbb{N}$ is coloured with one of two colours, B or Y, say. Then there is an infinite subset X of $\mathbb{N}$ all of whose two-element subsets have the same colour.*

Proof. We will specify an infinite sequence, $v_0, v_1, \ldots$, of natural numbers from which we can construct an infinite sequence of two-elements subsets $\{w_i, w_j\}$, all coloured B or all coloured Y. Let $V_0 = \mathbb{N}$ and let v_0 be its least element, i.e. 0. For all $i > 0$, $\{v_0, i\}$ is either B or Y. By the Pigeonhole Principle there is an infinite subset V_1 of $\mathbb{N}$ with $\{v_0, i\}$ having the same colour for all $i \in V_1$. Let v_1 be the least element of V_1 and repeat this process with V_1 in place of $\mathbb{N}$ to specify $v_2 \in V_2$ and then $v_3 \in V_3$, and so on. Now observe that, for any i, if $j > i$ then v_j lies in V_{i+1} so $\{v_i, v_j\}$ has the same colour. For short, let us say that v_i is either B or Y. This colours the whole sequence of v_is. Apply the Pigeonhole Principle again: there must be an infinite subsequence of the v_i having the same colour, say B. Write this sequence $w_0, w_1, \ldots$. Then every two-element subset $\{w_i, w_j\}$ of $\{w_0, w_1, \ldots\}$ is B.

Illustration. In the table above, B will be the odd-numbered columns and Y the even-numbered columns.

Take V_1 to be the odd numbers: all pairs $\{0, 2k+1\}, k \ge 0$, are B (odd). Then $v_1 = 1$. Take V_2 to be the alternate odd numbers: all pairs $\{1, 4k+3\}, k \ge 0$, are Y, and $v_2 = 3$. Take $V_3 = \{4k+3, k \ge 1\}$; all pairs $\{3, 4k+3\}, k \ge 1$, are B, etc.

A B-coloured subsequence: $w_0 = 0, w_1 = 3, w_2 = 7, w_3 = 11, \ldots$.

While the finite version of the Pigeonhole Principle has a long history, it is commonly attributed to Dirichlet who was most likely the first to use the infinite version in his *Lectures on Number Theory*, published posthumously in 1863.

Web link: www.cs.utexas.edu/ EWD/transcriptions/transcriptions.html: navigate to EWD980 (don't get sidetracked!)

Further reading: *Combinatorics and Graph Theory, 2nd edition* by John M. Harris, Jeffrey L. Hirst and Michael J. Mossinghoff Springer, 2008 (chapter 3). The example shown in the table is from chapter 7 of *Linear Orderings* by Joseph G. Rosenstein, Academic Press, 1982.