THEOREM OF THE DAY



The Pigeonhole Principle Let $q_1, q_2, ..., q_n$ be positive integers and let Q be a set of size $q_1 + ... + q_n - n + 1$.

Take a mapping from Q to a set $P = \{p_1, \ldots, p_n\}$ of size n. Then, under this mapping,

- 1. for some $i, 1 \le i \le n$, p_i is the image of at least q_i points in Q;
- 2. if any of the q_i are allowed to be infinity then some p_i in P is the image of infinitely many points in Q.

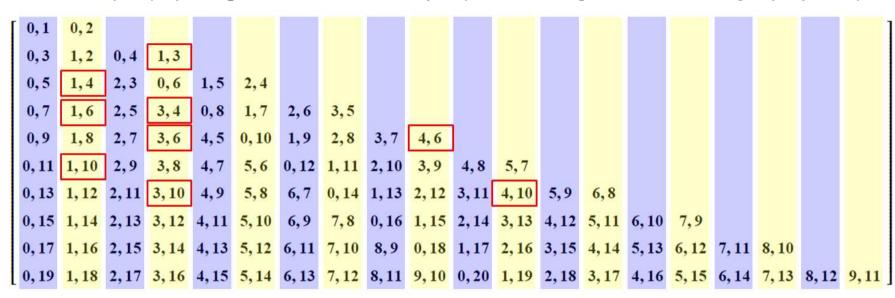


Table of unordered pairs $\{i, m-i\}, 0 \le i \le \lfloor (m-1)/2 \rfloor,$ with $m = 1 \dots n$. The table is infinite $(n \to \infty)$ but we have tabulated up to n = 20. Each table row covers two values of m, so the table has 10 rows. The pairs are partitioned as belonging in odd or even columns of the table. A challenge: find a large subset of numbers all of whose pairs lie in the even (yellow) columns. The boxed entries in the table perhaps look hopeful for the set {1, 3, 4, 6, 10}...

In fact, a set of five numbers can never have all pairs lying in even columns of the table! Symmetry might suggest a similar limit would hold for the odd columns as well but this is not true. We invoke an easy consequence of the infinite version of the Pigeonhole Principle, viz **The Infinite Ramsey Theorem** Suppose that each of the two-element subsets of \mathbb{N} is coloured with one of two colours, B or Y, say. Then there is an infinite subset X of \mathbb{N} all of whose two-element subsets have the same colour.

Proof. We will specify an infinite sequence, v_0, v_1, \ldots , of natural numbers from which we can construct an infinite sequence of two-elements subsets $\{w_i, w_j\}$, all coloured B or all coloured Y. Let $V_0 = \mathbb{N}$ and let v_0 be its least element, i.e. 0. For all i > 0, $\{v_0, i\}$ is either B or Y. By the Pigeonhole Principle there is an infinite subset V_1 of \mathbb{N} with $\{v_0, i\}$ having the same colour for all $i \in V_1$. Let v_1 be the least element of V_1 and repeat this process with V_1 in place of \mathbb{N} to specify $v_2 \in V_2$ and then $v_3 \in V_3$, and so on. Now observe that, for any i, if j > i then v_j lies in V_{i+1} so $\{v_i, v_j\}$ has the same colour. For short, let us say that v_i is either B or Y. This colours the whole sequence of v_i s. Apply the Pigeonhole Principle again: there must be an infinite subsequence of the v_i having the same colour, say B. Write this sequence w_0, w_1, \ldots Then every two-element subset $\{w_i, w_i\}$ of $\{w_0, w_1, \ldots\}$ is B.

Illustration. In the table above, *B* will be the odd-numbered columns and *Y* the even-numbered columns.

Take V_1 to be the odd numbers: all pairs $\{0, 2k + 1\}, k \ge 0$, are B (odd). Then $v_1 = 1$. Take V_2 to be the alternate odd numbers: all pairs $\{1, 4k + 3\}, k \ge 0$, are Y, and $v_2 = 3$. Take $V_3 = \{4k + 3, k \ge 1\}$; all pairs $\{3, 4k + 3\}, k \ge 1$, are B, etc.

A *B*-coloured subsequence: $w_0 = 0$, $w_1 = 3$, $w_2 = 7$, $w_3 = 11$,

While the finite version of the Pigeonhole Principle has a long history, it is commonly attributed to Dirichlet who was most likely the first to use the infinite version in his *Lectures on Number Theory*, published posthumously in 1863.

J. Larren J. Larren Web link: www.cs.utexas.edu/ EWD/transcriptions/transcriptions.html: navigate to EWD980 (don't get sidetracked!)

Further reading: Combinatorics and Graph Theory, 2nd edition by John M. Harris, Jeffrey L. Hirst and Michael J. Mossinghoff Springer, 2008 (chapter 3). The example shown in the table is from chapter 7 of Linear Orderings by Joseph G. Rosenstein, Academic Press, 1982.