THEOREM OF THE DAY

The Rotation Distance Bound *Any binary tree on n internal vertices, $n \geq 11$, may be transformed into any other using at most $2n-6$ rotations; and this bound is tight for sufficiently large n.*

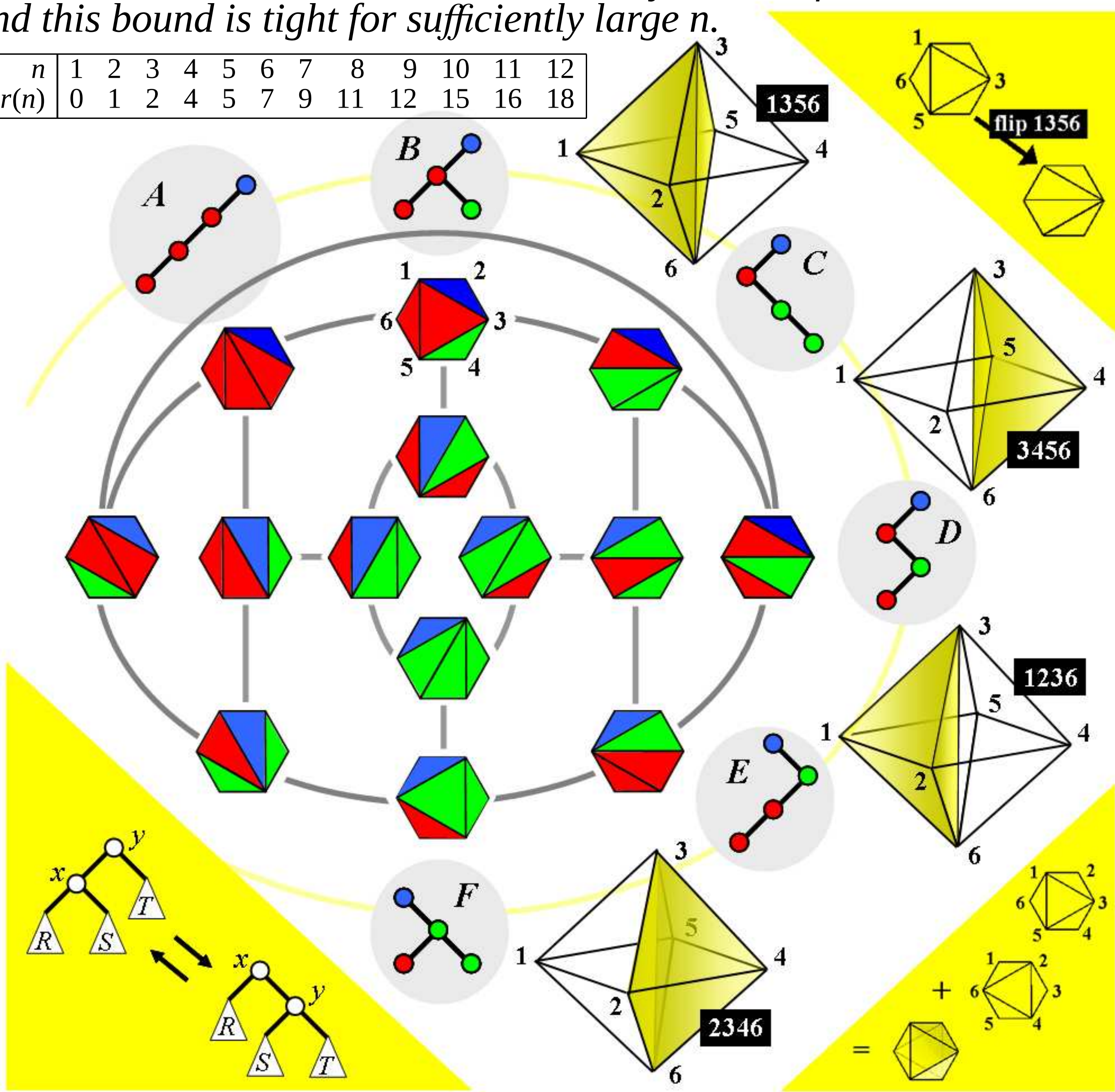
A binary tree consists of a *root vertex* joined to either zero or two *subtrees*, each of which is again a binary tree. Traditionally it is depicted as branching downwards with each vertex branching left (the *left child*) and right (*right child*). Vertices which have no subtrees are omitted from the drawing, in which only the *internal vertices* are shown. Six of the 14 distinct binary trees on four internal vertices are shown on the right, labelled *A*–*F*. The colours, red for left and green for right, are for illustrative purposes only.

Suppose a vertex y has left and right subtrees, with x being the root of the left subtree. A *rotation* at x moves x into y's place and y to the place of its right child, see the illustration, bottom left, in the yellow triangle. The left subtree of x and the right subtree of y (R and T, respectively, in the picture) are maintained, but the right subtree of x (i.e. S) becomes the left subtree of y. A rotation at the root of a right subtree, reversing the process, is also shown.

It is well-known that any binary tree may be transformed into any other on the same number of vertices by a series of rotations; thus B is transformed into F in the illustration; using, moreover, the least possible number of rotations. In 1988 Daniel Sleator, Robert Tarjan and William Thurston put an upper bound on the maximum distance $r(n)$, in number of rotations, between two trees with n internal vertices, by mapping such trees to triangulations of the $(n+2)$-gon. Trees A–F are shown adjacent to their corresponding triangulations here. Under the mapping rotations become 'flips': see the upper right yellow triangle; and some simple flip counting yields the upper bound $r(n) \leq 2n-6$.

Glueing together two polygons creates a polyhedron (an octagon in our case: bottom right triangle); a sequence of rotations becomes a 'triangulation' of the polyhedron (in our case using tetrahedra). And now an inspired switch from Euclidean to hyperbolic geometry produces, when n is sufficiently large, examples of triangulations which correspond to sequences of rotations meeting the $2n-6$ bound.

n	1	2	3	4	5	6	7	8	9	10	11	12
$r(n)$	0	1	2	4	5	7	9	11	12	15	16	18



Lionel Pournin proved in 2012 that this famous 1988 bound is in fact achieved for all $n \geq 11$, see gilkalai.wordpress.com/2012/12/30/.

Web link: hal.archives-ouvertes.fr/hal-00353917/. The original 1988 paper is here: www.cs.cmu.edu/~sleator/papers/Rotation-Distance.htm.

Further reading: T. Cormen, C. Leiserson, R. Rivest and C. Stein, Introduction to Algorithms, 3rd edition, MIT Press 2009, chapter 13.