THEOREM OF THE DAY

Strassen’s Matrix Theorem (a Theorem Under Construction!) Two \( n \times n \) matrices can be multiplied in fewer than \( n^3 \) (multiplication) steps.

Instructions for using this diagram

1. Entries of matrices \( A \) and \( B \) are arranged as shown. Now seven products are formed as follows:

2. Entries along curved lines (\( L_1, L_2 \) and \( L_7 \), passing four entries) and straight lines (\( L_3, L_4, L_5 \) and \( L_6 \), passing three) are added within each matrix and the sums are multiplied between the matrices. Line \( L_7 \) gives \( (a_{11} + a_{22}) \times (b_{11} + b_{22}) \), for example. But ...

3. ... where green (21) and blue (11) appear consecutively, the blue entry is subtracted rather than added; thus \( L_6 \) gives \( a_{22}(b_{21} - b_{11}) \); and ...

4. ... a red (22) consecutively with yellow (12) is likewise subtracted: \( L_4 \) gives \( (a_{12} - a_{22})(b_{21} + b_{22}) \).

5. Now \( AB = \left( \begin{array}{cc} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{array} \right) = \left( \begin{array}{cc} (L_1 - L_3 + L_6 + L_7) & (L_3 + L_4) \\ (L_5 + L_6) & (L_2 + L_4 - L_5 + L_7) \end{array} \right) \).

6. For larger matrices (padded out to size \( 2^k \times 2^k \) as necessary) partition into square blocks and proceed as above.

Construction notes: 1969: Volker Strassen amazes the world by multiplying \( n \times n \) matrices in \( O(n^{2.808}) \) (multiplication) steps. 1970s & 80s: Strassen’s result leads to a succession of lower values for the exponent, which becomes known as \( \omega \). 1987: Don Coppersmith and Shmuel Winograd achieve best value for \( \omega \) to date with \( O(n^{2.376}) \) algorithm. 2003: Chris Umans and Henry Cohn discover group theoretic formulation of Strassen’s approach. 2005: With Robert Kleinberg and Balazs Szegedy they reproduce Coppersmith and Winograd group theoretically and give two conjectures whose proof will reduce \( \omega \) to its theoretical minimum value of 2. 2011: Virginia Vassilevska Williams reduces \( \omega \) to 2.373. Meanwhile, Strassen’s original method remains the fastest practical way to multiply matrices larger than around 30 \( \times \)30.

Web link: an excellent overview: archive.siam.org/news/news.php?id=174; click the \( \text{\ding{253}} \) icon (top right) for news of the latest developments.