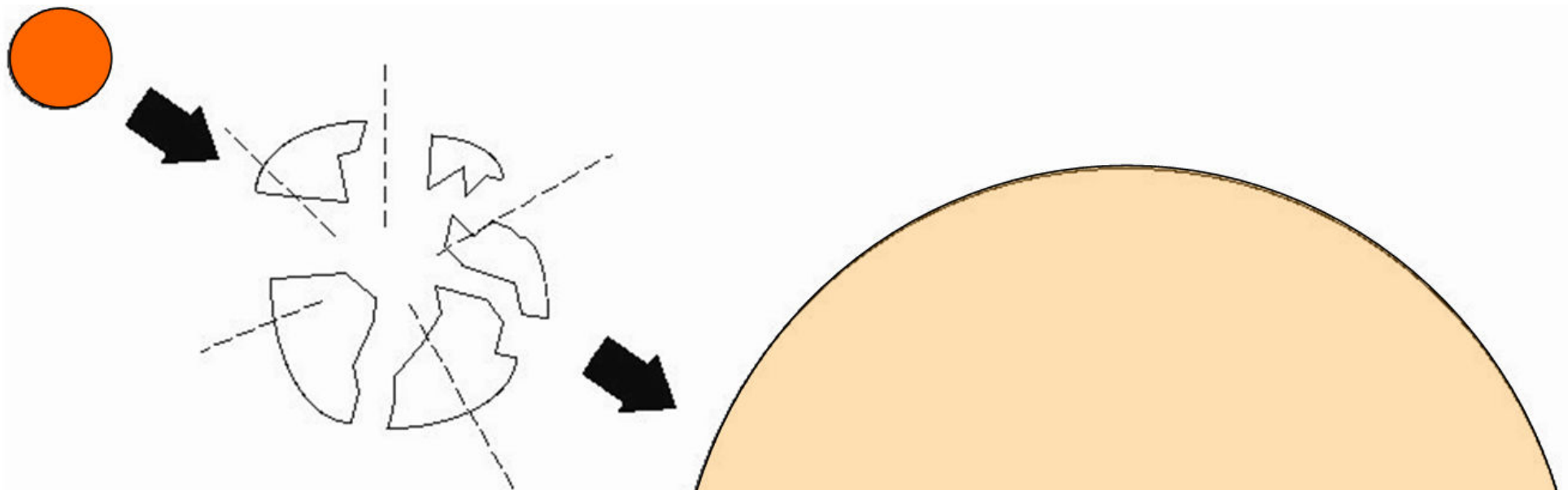




# THEOREM OF THE DAY



**The Well-ordering Theorem** Any set  $X$  can be well-ordered: an order relation 'less than' can be defined on  $X$  such that every non-empty subset of  $X$  contains a least element of  $X$ .



The well-ordering theorem can be used to solve all sorts of mathematical problems but it is a powerful medicine; from it follows the notorious **Banach-Tarski Paradox**: a solid sphere can be cut into five pieces which can be rearranged to form a solid sphere of twice (or any multiple) the original radius. The picture, showing an orange being reassembled to form the Sun, is a cheat since the five pieces must really be very complicated: they will make up a Sun with huge amounts of 'empty' space; but a mathematical orange has an infinity of material and infinity can fill up as much empty space as you need it to. At least it can if it is well-ordered!

Ernst Zermelo proved his theorem in 1904 using the **Axiom of Choice**: for any collection of nonempty sets there is a way of choosing a representative from each. It sounds innocent but, if an infinite number of sets is involved it is not at all clear how the set of representatives is to be specified; in fact, no general method is known and the axiom of choice remains that: an axiom. It and the well-ordering theorem are effectively equivalent to each other, so the theorem itself is treated as axiomatic but remains an indispensable tool in many parts of modern mathematics. It is essentially *non-constructive* since no general way of specifying the ordering is provided; even the positive real numbers have no known well-order; the usual ' $<$ ' will not work since the set of positive reals less than one, for example, has no least element under this order. But we can work on the basis, thanks to the well-ordering theorem, that the positive reals have been well-ordered 'by person or persons unknown'.

**Web link:** [plato.stanford.edu/entries/axiom-choice/](http://plato.stanford.edu/entries/axiom-choice/)

**Further reading:** *Mathematics: the Loss of Certainty* by Morris Kline, Oxford University Press, New York, 1980.

