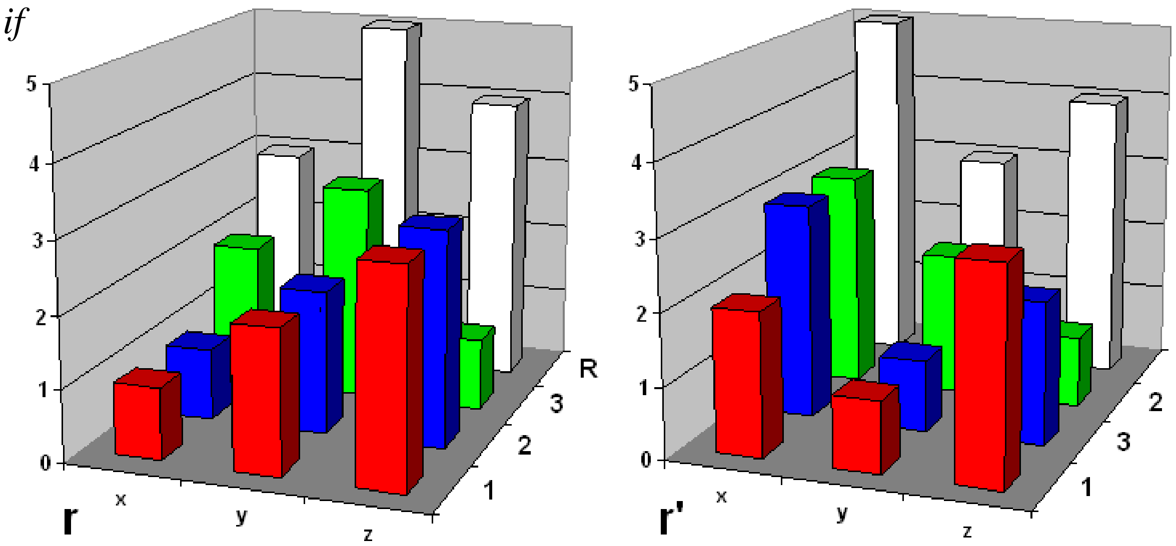
THEOREM OF THE DAY

Arrow's Impossibility Theorem *Let P be a set of m politicians and let $V = \{1, \ldots, n\}$, $n \geq 1$, be a set of voters. Let $\mathcal{R}$ be the set of all two-variable functions from $V \times P$ to $\{1, \ldots, m\}$ such that every $r \in \mathcal{R}$ is a ranking of the members of P, for each $v \in V$; that is, for each v, the values $r(v, p)$, $p \in P$, constitute a permutation of $\{1, \ldots, m\}$. Now suppose we have a* social choice function $R : \mathcal{R} \times P \to \{1, \ldots, m\}$ *which combines each two-variable ranking function $r \in \mathcal{R}$ with P to induce a one-variable ranking function from P to $\{1, \ldots, m\}$: for each $r \in \mathcal{R}$, the values $R(r, p)$ are again a permutation of $\{1, \ldots, m\}$. Then, if $|P| > 2$, our choice of R cannot satisfy all of the following three requirements for fair voting:*

Pareto Efficiency: *if everyone is unanimous about the respective merits of two politicians, then the social choice function should reflect this: for all $r \in \mathcal{R}$, if $r(v, p) > r(v, q)$ for all $v \in V$, then $R(r, p) > R(r, q)$;*

Independence from Irrelevant Alternatives (IIA): *if rankings r and r' agree on the relative merits of two politicians, say, p and q, then this should be reflected in the social choice function: if, for all $v \in V$, $r(v, p) > r(v, q)$ if and only if $r'(v, p) > r'(v, q)$, then $R(r, p) > R(r, q)$ if and only if $R(r', p) > R(r', q)$.*

Non-dictatorship: *no voter has the property that the social choice function always agrees with them regardless of what other voters do: there is no $v \in V$ for which $R(r, p) = r(v, p)$ for all $r \in \mathcal{R}$.*



On the right two members, r and r', of $\mathcal{R}$ are shown, for $P = \{x, y, z\}$ and $V = \{1, 2, 3\}$. The same social choice function R has been applied to both, displayed as the rear, white, bars. To define by example, the value of $R(r, z)$, the rightmost bar in the left-hand chart, was calculated by taking the product $r(1, z) \times r(2, z) \times r(3, z) = 3 \times 3 \times 1 = 9$; this was between the other two products, $1 \times 1 \times 2$ and $2 \times 2 \times 3$, so x, y and z were ranked 1st, 3rd and 2nd, respectively (represented here as 3, 5 and 4, to make them stand out). But this choice of R has violated IIA: r and r' agree, for 1, 2 and 3, on the relative merits of y and z, but $R(r, y) > R(r, z)$ while $R(r', y) < R(r', z)$.

Kenneth Arrow (Nobel prize for Economics, 1972) proved this hugely influential theorem in his 1951 PhD thesis.

Web link: derekbruff.org/voting/. The version of Arrow's Theorem given above is based on the relevant Wikipedia entry.

Further reading: *Game Theory and Its Applications in the Social and Biological Sciences,* by A.M. Colman, Routledge Falmer, 1995.