THEOREM OF THE DAY

Arrow's Impossibility Theorem *Let P be a set of m politicians and let V be a set of n voters, $m, n \geq 1$. A* **ballot** *is a list of n orderings of P which are strict (no ties) and transitive (if a voter ranks $p > q$ and $q > r$, for some $p, q, r \in P$, then that voter must rank $p > r$). A* **Social Welfare Function**, *S, maps each ballot to a strict, transitive ordering of P. Then S cannot satisfy all of the following:*

Unanimity : *if all voters rank $p > q$ then S must as well.*

Independence of Irrelevant Alternatives (IIA): *the ordering ($<$ or $>$) given by S to any pair of politicians is invariant under changes to all other pairwise comparisons, by voters, of members of P.*

Non-dictatorship: *No voter has the property that, for every ballot, their ordering of P is identical to the image of that ballot under S.*

Condorcet's Paradox: for three voters who make consistent (transitive) individual choices between candidates a majority decision may give an inconsistent result. For columns 5,11,18 below, the majority chooses $X > Y$ and $Y > Z$ but $Z > X$.

	X vs Y								Y vs Z								Z vs X							
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
A	>	>	>	>	<	<	<	<	>	>	>	>	<	<	<	<	>	>	>	>	<	<	<	<
B	>	>	<	<	>	>	<	<	>	>	<	<	>	>	<	<	>	>	<	<	>	>	<	<
C	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<
S	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<	>	<

Following a 'pedagogical proof' devised by Valentino Dardanoni, the essential ideas of Arrow's theorem and its proof may be presented as in the above table, with just three politicians, X, Y and Z and three voters, A, B and C. In this scenario, Unanimity is satisfied because S (row 4 in the table) is identical to the unanimous choices in columns 1,8,9,16,17 and 24. IIA is satisfied in the strong sense that the value of S is specified column by column. Therefore no change of column in any two parts of the table can affect the third part. But row C violates Non-dictatorship: it is identical to S. Suppose, to illustrate, that C initially joined the unanimous vote in Column 17, preferring Z to X; S necessarily agrees by Unanimity. But now C decides they prefer X to Z, switching to Column 18. S switches too, siding with C against the majority! The inevitability of this conflict is demonstrated by Dardanoni as follows: (1) he shows that if, in any column of the table, S disagrees with a majority decision (as in column 18, for example), then the voter in the minority must be a dictator; and (2) he observes that, for three columns constituting a Condorcet paradox (see box above right), either S always agrees with the majority and is then non-transitive, or S is transitive but then we are forced us to be in case (1), producing a dictator.

Arrows ideas, set out in the 1951 book *Social Choice and Individual Values*, gave birth to a whole branch of economics, Social Choice Theory, and earned him the Nobel Prize in 1972.

Web link: derekbruff.org/voting/.

Further reading: *Game Theory and Its Applications in the Social and Biological Sciences,* by Andrew M. Colman, Routledge Falmer, 1995.