Green’s Theorem  
Let $C$ be a closed, anticlockwise-oriented curve in the $xy$-plane enclosing a region $D$. Let $F(x,y) = (P(x,y), Q(x,y))$ be a 2-valued function having continuous partial derivatives on $C$ and inside $D$. Then

$$\int_C F \, ds = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$ 

Suppose we think of $F$ as a force field acting in the plane. Then the line integral $\int_C F \, ds$ may be thought of as measuring the total work done by $F$ acting on a particle as it follows the curve $C$; the integral is often written in the form $\int_C P \, dx + Q \, dy$, making explicit the action of the $x$ and $y$ components of the force as the particle moves through a small increment in the $x$ and $y$ directions. We refer to this work done by $F$ around $C$. Green’s Theorem asserts that circulation around $C$ is the accumulation of ‘microscopic circulations’ around points in $D$: see the illustration on the left; these microscopic circulations are measured as the component perpendicular to the plane of the curl of $F$; this component is calculated as: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

As an illustration we take $P(x,y) = (x - y)^2$ and $Q(x,y) = x + y$. These are plotted as surfaces in 3D on the right, and a half-unit circle, closed by adjoining a segment of the $x$-axis, is chosen as the curve $C$. We find $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 + 2(x - y)$ and Green’s Theorem yields the value of $\int_C F \, ds$ by double integration as

$$\int \int_D 1 + 2(x - y) \, dA = \int_0^1 \int_0^{\sqrt{1-y^2}} 1 + 2(x - y) \, dy \, dx = \int_0^1 \left( \sqrt{1-x^2} + 2x \sqrt{1-x^2} - 1 + x^2 \right) \, dx = \left[ \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \frac{2}{3} (1-x^2)^{3/2} - x + \frac{x^3}{3} \right]_0^1 = \frac{\tau}{4} - \frac{4}{3}.$$ 

Evaluating the line integral directly requires piecewise parameterisation of the curve $C$: half-circle $c_1(t) = (-\cos t, \sin t), 0 \leq t \leq \tau/2$ and base $c_2(t) = (t, 0), -1 \leq t \leq 1$:

$$\int_C F \, ds = \int_0^{\tau/2} F(c_1(t)) \cdot c_1'(t) \, dt + \int_{-1}^{1} F(c_2(t)) \cdot c_2'(t) \, dt = \int_0^{\tau/2} F(\cos t, \sin t) \cdot (-\sin t, \cos t) \, dt + \int_{-1}^{1} F(t, 0) \cdot (1,0) \, dt$$

$$= \int_0^{\tau/2} (-\cos t - \sin t)^2(-\sin t) + (\cos t + \sin t) \cos t \, dt + \int_{-1}^{1} t^2 \, dt = -2 + \frac{\tau}{4} + \frac{2}{3} = \frac{\tau}{4} - \frac{4}{3},$$

as expected.

George Green published this theorem, a powerful generalisation of the Fundamental Theorem of the Calculus, in 1828.

Web link: mathinsight.org/greens_theorem_idea (on which the above description and examples are based).