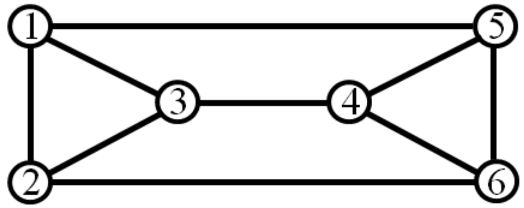
THEOREM OF THE DAY

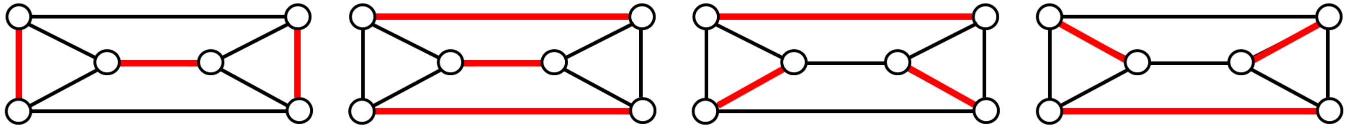
Kasteleyn's Theorem *Suppose that G is a planar graph drawn in the plane. Then*

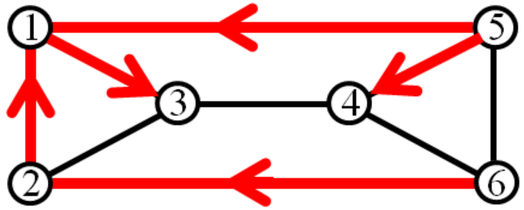
1. *we can orient the edges so that every face has an odd number of clockwise-oriented edges, and*
2. *if $A(G)$ is the signed adjacency matrix of the orientation of G then*

$$\textit{number of perfect matchings of } G = \sqrt{\det(A(G))}.$$



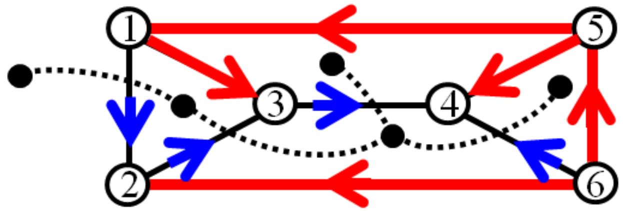
A perfect matching is set of disjoint edges which includes every vertex. The graph top right, for example, has four perfect matchings, as shown below.





The example on the right illustrates how part (1) of Kasteleyn's theorem is proved.

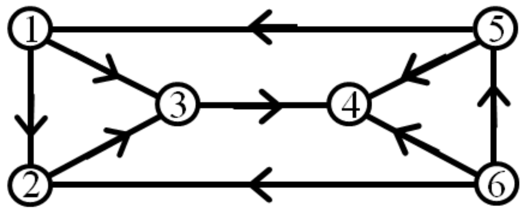
1. Take a spanning tree T of G and orient its edges arbitrarily (the heavy, red edges in the 2nd drawing on the right).
2. Take the dual graph (vertices are faces of G, edges cross edges of G from face to face). In the dual, take a spanning tree T^* whose edges cross edges of G that are not in T (the dotted edges in the bottom drawing).
3. Starting at leaves of T^* located in internal faces, orient the G edges crossed by T^* edges so as to give each face an odd number of clockwise-oriented edges (the heavy, blue arrows in the bottom drawing).



The faces of our graph, in the order in which they are oriented, end up with the following numbers of clockwise edges: 1543:**1**; 456:**1**; 2346:**3**; 132:**1**. As a check, the outside edge should also be 'odd': 1562:**1**;.

Our example has resulted in an oriented version of the original graph G as shown, far right. Its signed adjacency matrix is shown near right: the entry in the ith row and jth column is set to +1 to record an edge oriented from i to j; and −1 for an edge oriented from j to i. The determinant function, calculated rapidly by a spreadsheet or mathematics package, has value 16.

$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} & 1 & 1 & & -1 & \\ -1 & & 1 & & & -1 \\ -1 & -1 & & 1 & & \\ & & -1 & & -1 & -1 \\ 1 & & & 1 & & -1 \\ & 1 & & 1 & 1 & \end{pmatrix} \end{array}$$



Enumeration of perfect matchings has important applications in statistical physics but is computationally infeasible in the general case. The enumeration equates to a generalisation of the determinant function discovered by Johann Friedrich Pfaff in 1815 and dubbed the *Pfaffian* by Cayley. For the $m \times n$ grid, Pieter Willem Kasteleyn and, independently, HNV Temperley and Michael Fisher showed in 1961 how a skew-symmetric version of the adjacency matrix can be adopted, guaranteeing (Cayley, 1847) that the Pfaffian is the square of the easily computed determinant. Kasteleyn's approach generalised to his celebrated 1967 theorem for arbitrary planar graphs, presented here.

Web link: www.ams.org/notices/200503/: the "WHAT IS...?" article of Richard Kenyon and Andrei Okounkov

Further reading: Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra by Jiří Matoušek, AMS, 2010.