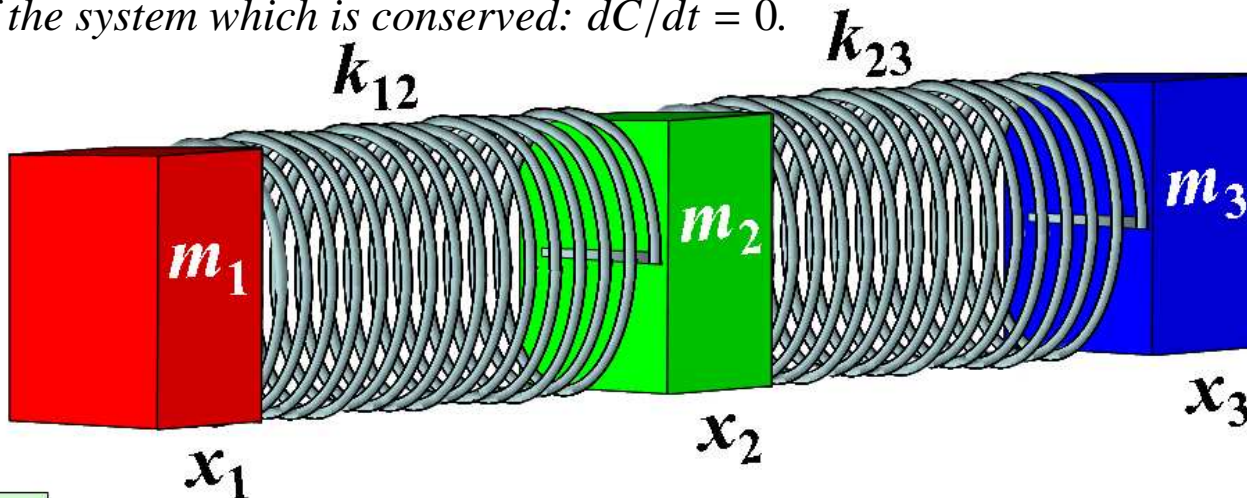




THEOREM OF THE DAY

Noether's Symmetry Theorem Suppose a system of particles in classical mechanics exhibits some symmetry, i.e. its Lagrangian, \mathcal{L} , is invariant under changes to some variable s , so that $\partial\mathcal{L}/\partial s = 0$. Then there is some associated property C of the system which is conserved: $dC/dt = 0$.

$$\mathcal{L}(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) = T - V$$



$$\text{Kinetic energy } T = \frac{1}{2}(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2)$$

$$\text{Potential energy } V = \frac{1}{2}(k_{12}(x_2 - x_1)^2 + k_{23}(x_3 - x_2)^2)$$

The behaviour of a system of particles in classical mechanics may be described using the Lagrangian, \mathcal{L} , the difference between its kinetic and potential energies as calculated from position and velocity parameters, q and \dot{q} . (In the above one-dimensional system of massless springs, the parameters are x_i and \dot{x}_i , $i = 1, \dots, 3$, with k_{ij} being the spring coefficients and m_i , the masses.) Newton's laws of motion are then embodied in the Euler-Lagrange equation: $\partial\mathcal{L}/\partial q = d/dt(\partial\mathcal{L}/\partial\dot{q})$. Under the belief that mechanical laws are invariant over time t , we assert that $\partial\mathcal{L}/\partial t = 0$, so that changes in \mathcal{L} over time depend purely on changes to its parameters:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) = \sum_i \left(\dot{q}_i \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{q}_i} + \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) \text{ (substituting in the first term via the Euler-Lagrange equation).}$$

Rearranging and expressing as a single time derivative, we have $\frac{d}{dt} \left(\mathcal{L} - \sum_i \dot{q}_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \right) = 0$, with the expression in the bracket being a conserved property of the system. Evaluating this bracketed expression for the spring system shown above we get $\mathcal{L} - 2T$, and since $\mathcal{L} = T - V$, we find that the property being conserved is (minus) $T + V$, the total energy: *time invariance* \leftrightarrow *conservation of energy*.

Emmy Noether's theorem, proved shortly after she took up a post (unofficial and unpaid) at Göttingen University in 1915, is a profound reinterpretation of the Euler-Lagrange equation. It extends to quantum mechanical systems and now underlies the Standard Model of modern particle physics. Just as in the above example, new invariance properties are investigated via the theorem to identify the conservation laws which they entail.

Web link: www.mathpages.com/home/kmath564/kmath564.htm (I adapted the spring example from this admirable explanation).

Further reading: *Emmy Noether's Wonderful Theorem* by Dwight Neuenschwander, The Johns Hopkins University Press, 2010.

