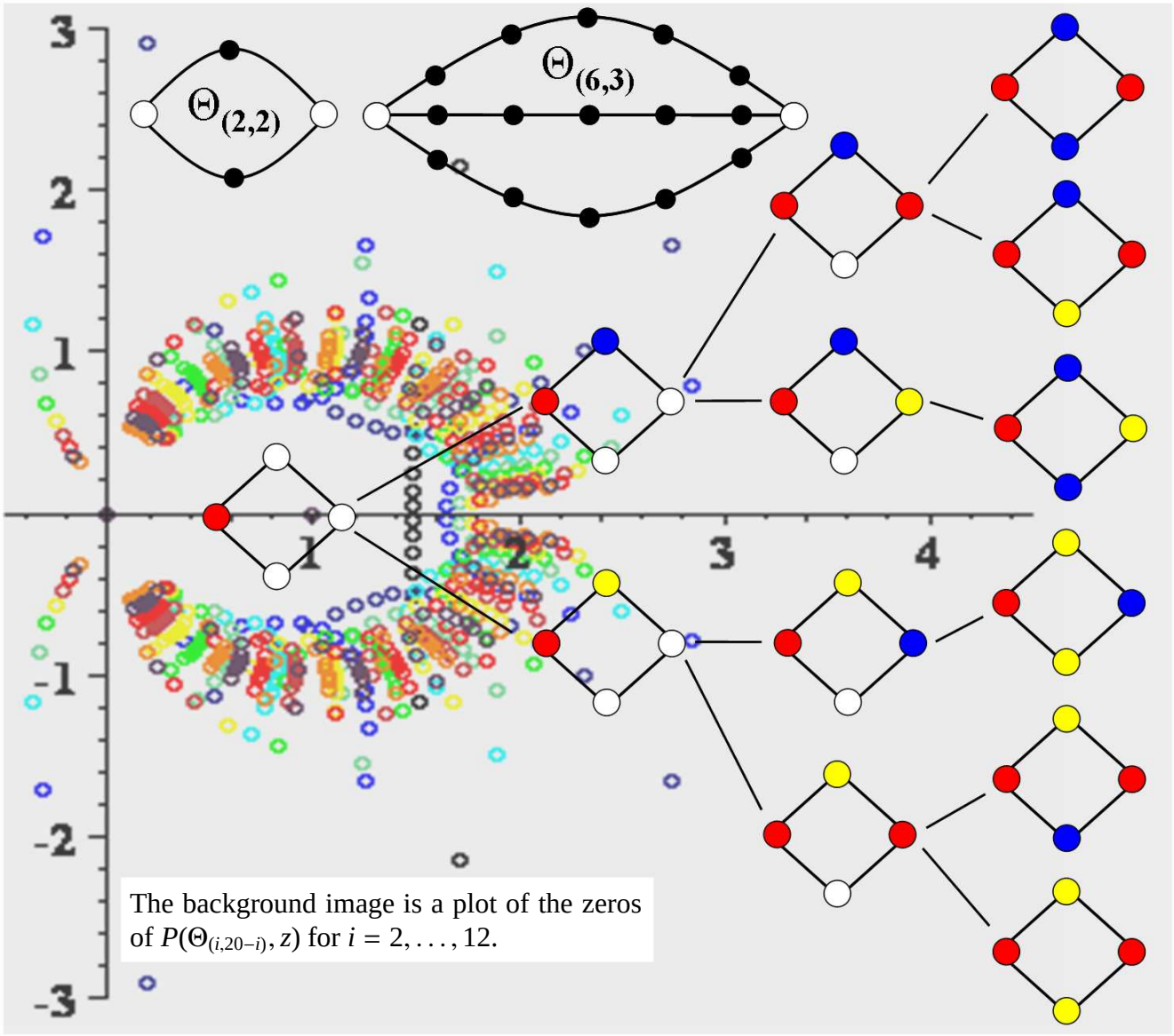
THEOREM OF THE DAY

Sokal's Theorem on Chromatic Roots *The roots of chromatic polynomials of graphs, taken together, are dense in the whole complex plane.* *(... in fact, except for a unit disc around $z = 1$, it is enough to take the generalised theta graphs $\Theta_{(r,s)}$, which consist of s paths of length r joining two end vertices.)*



The background image is a plot of the zeros of $P(\Theta_{(i,20-i)}, z)$ for $i = 2, \ldots, 12$.

The proof of this celebrated theorem in 2000 by Alan Sokal marked an exciting collision between combinatorics and mathematical physics.

The cycle of length 4, for example, is a generalised theta graph, namely $\Theta_{(2,2)}$. It has chromatic polynomial

$$P(\Theta_{(2,2)}, z) = -3z + 6z^2 - 4z^3 + z^4,$$

defined as the unique polynomial whose value at $z = k$, for a nonnegative integer k, is the number of ways to vertex-colour the graph using at most k colours so that no adjacent vertices have the same colour. The polynomial evaluates to zero at $z = 1$ because we cannot properly colour any graph with just one colour unless it has no edges. The value for $z = 3$ is 18 and we verify that this is correct, in our illustration, by enumerating six possible 3-colourings (i.e. from a palette of 3 colours) which start with a red vertex on the left (there are then 6×3 total colourings by replacing red with blue or yellow). Another zero is $z = 0$ (no colourings possible with an empty palette!); and there are two complex roots: $z = 3/2 \pm i\sqrt{3}/2$; their meaning in terms of graph colouring is obscure, but they are of great interest to mathematical physicists, who interpret them in terms of phase transitions in physical systems.

Even for a graph which is still very small, $\Theta_{(6,3)}$, the chromatic polynomial becomes a fearsome beast:

$$\begin{aligned} &91\,z - 711\,z^2 + 2955\,z^3 - 8505\,z^4 + 18543\,z^5 - 31821\,z^6 + 43758\,z^7 \\ &\quad - 48620\,z^8 + 43758\,z^9 - 31824\,z^{10} + 18564\,z^{11} \\ &\quad - 8568\,z^{12} + 3060\,z^{13} - 816\,z^{14} + 153\,z^{15} - 18\,z^{16} + z^{17} \end{aligned}$$

It has 17 zeros with $z \approx 2.066 - .311i$ being one with largest real part. The graph has only two 2-colourings: $P(\Theta_{(6,3)}, 2) = 2$; but with three colours we can colour in 87510 ways: $P(\Theta_{(6,3)}, 3) = 87510$; and with four colours the value is close to 10^8.

For graph theorists this theorem is, in a sense, a negative result: for zeros of planar graph chromatic polynomials to be dense in the complex plane means that we can find, in an arbitrarily small disc around $(4, 0)$, a zero of $P(G, z)$ for some planar graph G. So it seems hopeless to appeal to complex variable theory for a proof of the Four-Colour Theorem ($P(G, 4) > 0$ for any planar G).

Web link: arxiv.org/abs/math/0205047

Further reading: *Algebraic Graph Theory (2nd Edition)* by Norman Biggs, Cambridge University Press, 1994.