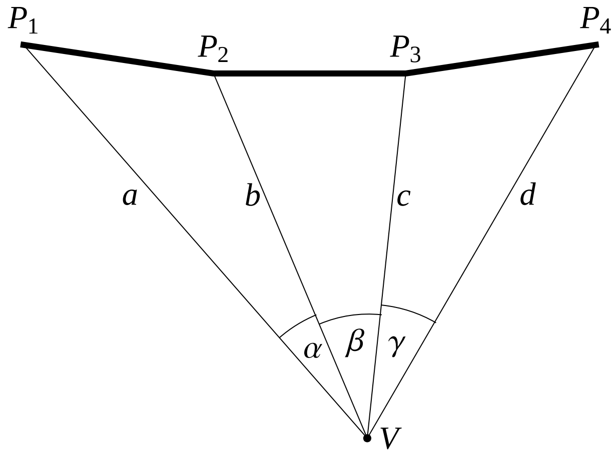
Solution 316.5 – Prism

This is like M500 Problem 315.6 – Prism in that you are to determine from limited data the n of a regular n-sided prism. Suppose you are standing at point V outside the prism such that four consecutive corners P_1, P_2, P_3 and P_4 are visible. Let

$$\alpha = \angle P_1VP_2, \quad \beta = \angle P_2VP_3, \quad \gamma = \angle P_3VP_4.$$

Compute n as a function of α, β and γ.



Tony Forbes

This is not a solution I am totally happy with. Let

$$1 = |P_1P_2|, \quad a = |P_1V|, \quad b = |P_2V|, \quad c = |P_3V|, \quad d = |P_4V|.$$

After making the sensible definitions

$$\mu = \frac{\pi}{n}, \qquad m = 2\cos\mu,$$

so that

$$\begin{aligned}\angle P_1P_2P_3 &= \angle P_2P_3P_4 = \pi - 2\mu,\\ |P_1P_3| &= |P_2P_4| = m,\\ |P_1P_4| &= m^2 - 1,\end{aligned}$$

an obvious way to start is by writing down

$$
\begin{aligned}
1 &= a^2 + b^2 - 2ab\cos\alpha, && (1)\\
1 &= b^2 + c^2 - 2bc\cos\beta, && (2)\\
1 &= c^2 + d^2 - 2cd\cos\gamma, && (3)\\
m^2 &= a^2 + c^2 - 2ac\cos(\alpha+\beta), && (4)\\
m^2 &= b^2 + d^2 - 2bd\cos(\beta+\gamma), && (5)
\end{aligned}
$$

five equations in five unknowns, a, b, c, d and m. We use (1), (2) and (5) to eliminate a, c and d by

$$
\begin{aligned}
a &= b\cos\alpha + B(\alpha), && (6)\\
c &= b\cos\beta + B(\beta), && (7)\\
d &= b\cos(\beta+\gamma) + \sqrt{m^2 - b^2\sin^2(\beta+\gamma)}, && (8)
\end{aligned}
$$

where we have defined

$$B(\theta) = \sqrt{1 - b^2\sin^2\theta}.$$

Now we only have to worry about b and m. Moreover, we can obtain m in terms of b from (4), (6) and (7):

$$
\begin{aligned}
m^2 = {} & (b\cos(\alpha) + B(\alpha))^2 + (b\cos(\beta) + B(\beta))^2 \\
& - 2\cos(\alpha+\beta)\,(b\cos(\beta) + B(\beta))\,(b\cos(\alpha) + B(\alpha)).
\end{aligned} \qquad (9)
$$

It is straightforward if a little tedious to take equality (3) and substitute c from (7), d from (8) and m^2 from (9) to get a single equation involving only the length b:

$$
\begin{aligned}
1 = {} & (b\cos(\beta) + B(\beta))^2 + \left(b\cos(\beta+\gamma) + \sqrt{\Delta}\right)^2 \\
& - 2\cos(\gamma)\left(b\cos(\beta+\gamma) + \sqrt{\Delta}\right)(b\cos(\beta) + B(\beta)),
\end{aligned} \qquad (10)
$$

where

$$
\begin{aligned}
\Delta = {} & (b\cos(\alpha) + B(\alpha))^2 + (b\cos(\beta) + B(\beta))^2 - b^2\sin^2(\beta+\gamma) \\
& - 2\cos(\alpha+\beta)\,(b\cos(\beta) + B(\beta))\,(b\cos(\alpha) + B(\alpha)).
\end{aligned}
$$

The next step is to solve (10) for b. Unfortunately that appears to be rather difficult. If anyone disagrees, they are welcome to try.

It seems that any attempt to obtain an exact formula for n as a function of the three angles is going to be doomed. Nevertheless, just to reassure oneself that one's analysis is at least approximately correct, let us examine a few special cases in a non-exact manner.

We assume our surveyor has made accurate measurements in degrees of angles α, β and γ. We plug these into (10) and then use numerical methods to determine b. Substituting α, β, γ and b into (9) gives m^2, and finally we can determine n from

$$n \;=\; \frac{\pi}{\arccos(m/2)}. \tag{11}$$

For consistency, we give this integer to 6 significant figures. You might like to enjoy yourself by working through the process manually to confirm the entries in the table. If you set your calculator to degrees, don't forget to substitute 180 for π in (11).

α	β	γ	b	m	n
1.94857	14.3622	2.36779	4.00828	1.61803	5.00000
6.88565	10.8013	1.03324	4.89045	1.73205	6.00000
4.24431	5.23243	1.54748	10.2764	1.80194	7.00000
8.47430	17.5212	7.97913	3.27283	1.84776	8.00000
6.28818	6.27499	2.41115	8.19475	1.87939	9.00000
7.25745	10.1009	6.32791	5.63547	1.90211	10.0000
9.90571	8.64602	3.76179	5.59375	1.93185	12.0000
5.36626	5.66085	4.13542	9.84587	1.94986	14.0000
5.49091	5.33704	3.83864	10.1422	1.96157	15.9999
7.46110	8.60031	7.21108	6.65272	1.96962	18.0000
10.3737	11.6855	9.17777	4.84862	1.97538	19.9997
5.59791	5.19166	4.11562	10.2369	1.98423	24.9998
8.35400	8.64687	7.47176	6.53538	1.98904	29.9998
11.7302	10.4056	7.60657	4.90206	1.99195	34.9990
5.74654	5.52637	4.86054	9.96383	1.99383	39.9990
7.19626	7.12247	6.37140	7.86941	1.99513	44.9993
7.51561	7.31167	6.46818	7.58859	1.99605	50.0014
5.80994	5.58789	5.06003	9.87536	1.99726	60.0005
12.3631	11.1280	8.82155	4.66487	1.99799	69.9936
17.1436	14.0396	9.80238	3.39193	1.99846	79.9994
9.88948	10.3368	9.73035	5.53952	1.99878	89.9974
8.18650	8.38253	7.98001	6.81809	1.99901	99.9989

A gravitationally bound atom

Tony Forbes

What is the radius of a negatively charged 'atom' with atomic number 0 that consists of a neutron and an electron bound to it by gravity?

In an attempt to answer this question, we shall assume (rather naïvely I suspect) that we have here the electron in a circular orbit around the neutron. Then we invoke quantum theoretical principles by insisting that the angular momentum of the electron is limited to positive integer multiples of $\hbar$, the reduced Planck's constant. We ignore spin. We also ignore the power dissipated by the accelerating electron ($< 10^{-243}$ W).

For an orbit of radius r, by equating forces whilst noting the absence of an electromagnetic component, we have

$$\frac{m_{\mathrm{n}} m_{\mathrm{e}} G}{r^2} = m_{\mathrm{e}} r \omega^2, \tag{1}$$

where ω is the electron's angular velocity. Denote the electron's angular momentum by L and its ordinary momentum in the direction tangential to the orbit by p. Then

$$L = m_{\mathrm{e}} r^2 \omega = n\hbar \quad \text{and} \quad p = \frac{L}{r}. \tag{2}$$

We can solve (1) and (2) to obtain

$$r = \frac{n^2 \hbar^2}{m_{\mathrm{n}} m_{\mathrm{e}}^2 G}. \tag{3}$$

To verify that this is a sensible conclusion, we can compute the wavelength of the electron from the formula $\lambda = h/p$:

$$\lambda = \frac{h}{p} = \frac{hr}{L} = \frac{hr}{n\hbar} = \frac{2\pi r}{n},$$

which is indeed an integer fraction of the circumference of the orbit.

Substituting actual values in (3) for the constants,

$$m_{\mathrm{n}} = 1.67492750056 \times 10^{-27}\,\mathrm{kg}, \quad m_{\mathrm{e}} = 9.1093837139 \times 10^{-31}\,\mathrm{kg},$$
$$h = 2\pi\hbar = 6.62607015 \times 10^{-34}\,\mathrm{J\,s}, \quad G = 6.67430 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2},$$

for which we acknowledge help from *Wikipedia*, we obtain

$$r_1 = 1.19887 \times 10^{29}\,\mathrm{m} = 1.26721 \times 10^{13}\,\mathrm{ly}.$$

for the ground state, $n = 1$. Since r_1 is rather large we probably won't be able to detect gravitationally bound neutron–electron atoms until the Universe has thinned out a little. Moreover, we expect these atoms to decay to hydrogen-1 with the same half-life as that of a free neutron, about 10 minutes ($\mathrm{n} \to \mathrm{p} + \mathrm{e} + \overline{\nu}_{\mathrm{e}}$, occasionally $\mathrm{n} \to \mathrm{p} + \mathrm{e} + \overline{\nu}_{\mathrm{e}} + \gamma$ [*Wikipedia*]).

One is tempted to go on to study the atom's spectrum. However, I appreciate that some would regard this as a futile exercise. If the atom were created in the first excited state, $n = 2$, then a jump to the ground state, $n = 1$, would involve a change of orbit radius amounting to about 3.8×10^{13} ly. The neutron would almost certainly decay long before the completion of the transition.

The total energy (potential + kinetic) of the electron is

$$E_n \;=\; -\int_r^{\infty} \frac{m_{\mathrm{n}} m_{\mathrm{e}} G}{s^2}\,ds + \frac{m_{\mathrm{e}}(r\omega)^2}{2} \;=\; -\frac{m_{\mathrm{n}} m_{\mathrm{e}} G}{2r} \;=\; -\frac{m_{\mathrm{n}}^2 m_{\mathrm{e}}^3 G^2}{2n^2\hbar^2}.$$

Therefore we can calculate the energy difference between orbits n and n':

$$E_{n'\to n} \;=\; E_{n'} - E_n \;=\; -\frac{m_e}{2}\left(\frac{m_{\mathrm{n}} m_{\mathrm{e}} G}{\hbar}\right)^2 \left(\frac{1}{(n')^2} - \frac{1}{n^2}\right). \qquad (4)$$

The expression $m_{\mathrm{n}} m_{\mathrm{e}} G/\hbar \approx 9.7 \cdot 10^{-34}$ m/s is the electron's speed in the ground state. The energy difference when going from level 2 to level 1 is

$$E_{2\to 1} \;=\; E_2 - E_1 \;\approx\; 3.2 \times 10^{-97}\,\mathrm{J}.$$

If by some miracle the atom remains intact throughout the transition, the wavelength of the ultra-long-wave radio photon emitted would be

$$\lambda_{2\to 1} \;=\; \frac{hc}{E_{2\to 1}} \;=\; 6.6 \times 10^{55}\,\mathrm{ly}$$

approximately.

Finally, one might like to compare (4) with the corresponding formula—also provided by *Wikipedia*—for the hydrogen atom, one proton, one electron and no gravity:

$$H_{n'\to n} \;=\; -\frac{m_{\mathrm{e}}}{2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2 \left(\frac{1}{(n')^2} - \frac{1}{n^2}\right).$$

Here, the wavelength for the transition $H_{2\to 1}$ is a mere 1.285×10^{-23} ly, somewhere in the ultraviolet.

Bomb delivery

Tony Forbes

Suppose you wish to drop a bomb from an aircraft flying at great height. If the bomb is powerful, like the 50 Mt Tsar Bomba, you and the aircraft do not want to be too near the thing when it explodes.

One obvious escape strategy is to turn the aircraft through 180 degrees and fly directly away from the target. Surprisingly to me, but probably not to bomber pilots, this simple procedure is not optimal. In what follows we offer a better escape path based on a few simplifying assumptions.

Until it is far from the target, assume the aircraft's speed is a constant, v, and that its height above the ground is also constant.

Assume the bomb is released from $(0, 0)$ in the (x, y)-plane and detonates after time t at (x, y)-location

$$D = (0,\ \alpha v t),$$

where $\alpha \le 1$ is a factor to allow for air resistance slowing the bomb down from the horizontal speed v which it acquires from the aircraft.

After releasing the bomb the aircraft follows a circular path of radius r for angle θ, $\pi/2 \le \theta \le \pi$ to get to point

$$A = (r - r\cos\theta,\ r\sin\theta).$$

It then flies straight for a distance

$$f = vt - r\theta, \qquad vt > r\theta,$$

to arrive at

$$B = A + (f\sin\theta,\ f\cos\theta),$$

when the bomb explodes. Look at the diagram on the next page, and it might help to recall that

$$\cos(\pi - \theta) = -\cos\theta \quad \text{and} \quad \sin(\pi - \theta) = \sin\theta.$$

We wish to maximize the distance from the explosion. Equivalently, we maximize $|B - D|^2$ as a function of θ. We have

$$B - D = (r - r\cos\theta + f\sin\theta,\ r\sin\theta + f\cos\theta - \alpha v t)$$

and hence

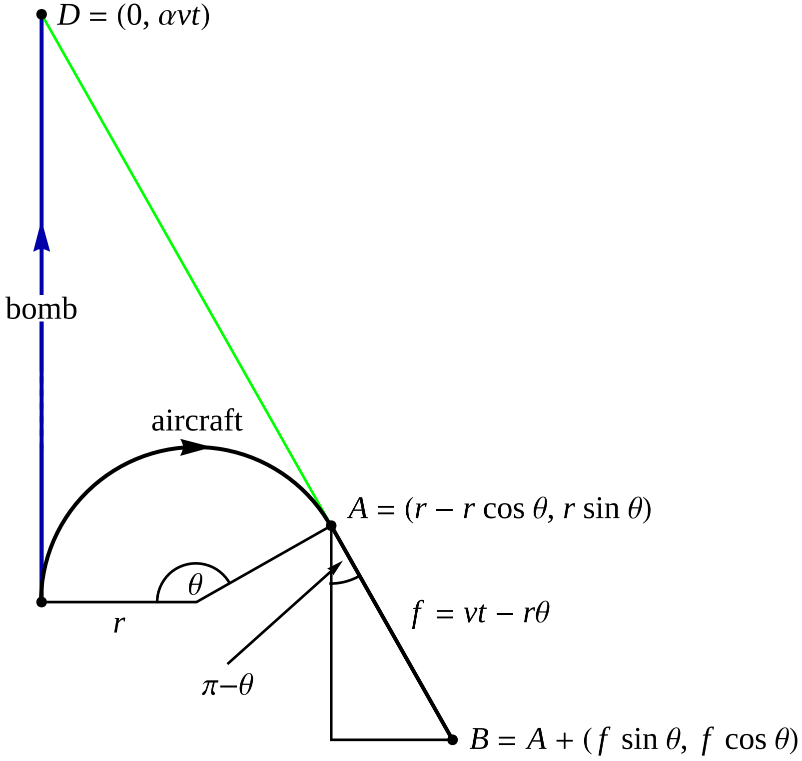
$$|B - D|^2 = (r - r\cos\theta + f\sin\theta)^2 + (r\sin\theta + f\cos\theta - \alpha v t)^2\,.$$

Differentiating and simplifying gives

$$\frac{d|B-D|^2}{d\theta} = 4(r\theta - tv)\sin\left(\frac{\theta}{2}\right)\left(r\sin\left(\frac{\theta}{2}\right) - \alpha tv\cos\left(\frac{\theta}{2}\right)\right).$$

The only point consistent with the constraints on θ where this conveniently factorized expression vanishes occurs at

$$\theta_0 = 2\arccos\left(\frac{r}{\sqrt{r^2+\alpha^2t^2v^2}}\right). \tag{1}$$



Having solved the problem according to the traditional method of locating the maximum of the distance function by differentiating and equating to zero, we now see that there has to be an easier way. The diagram clearly shows that D, A, and B are collinear. So it turns out that (1) can be determined by elementary trigonometry.