Pythagoras' theorem revisited

Tony Forbes

Students Calcea Johnson and Ne'Kiya Jackson of St Mary's Academy in New Orleans have generated a certain amount of excitement amongst the mathematical community by proving Pythagoras' theorem in a new and interesting manner. See Leila Sloman's article of 10 April 2023 in *Scientific American*, which is available at

`https://www.scientificamerican.com/article/2-high-school-students-prove-pythagorean-theorem-heres-what-that-means/`.

Unfortunately no details of the proof were given.

At a London South Bank University Maths Study Group meeting in April 2023 we were shown a video claiming to explain how the two students might have proved the theorem. However, I found the long-winded presentation rather tedious and the temptation to get some sleep became almost irresistible. Nevertheless I managed to salvage the main features of the lecturer's diagram from which a proof is quite easy to concoct.

Here is a proof based on the first picture on the next page. I do not know if this is actually how the students did it. All the relevant triangles are similar, and it helps if $b > a$. We have

$$\begin{aligned} |B_1B_2| &= \frac{2ca}{b}, & |D_1B_2| &= \frac{2a^2}{b}, & |D_1D_2| &= \frac{2ca^2}{b^2}, \\ |B_2B_3| &= \frac{2ca^3}{b^3}, & |D_2B_3| &= \frac{2a^4}{b^3}, & |D_2D_3| &= \frac{2ca^4}{b^4}, \\ &\ldots. \end{aligned}$$

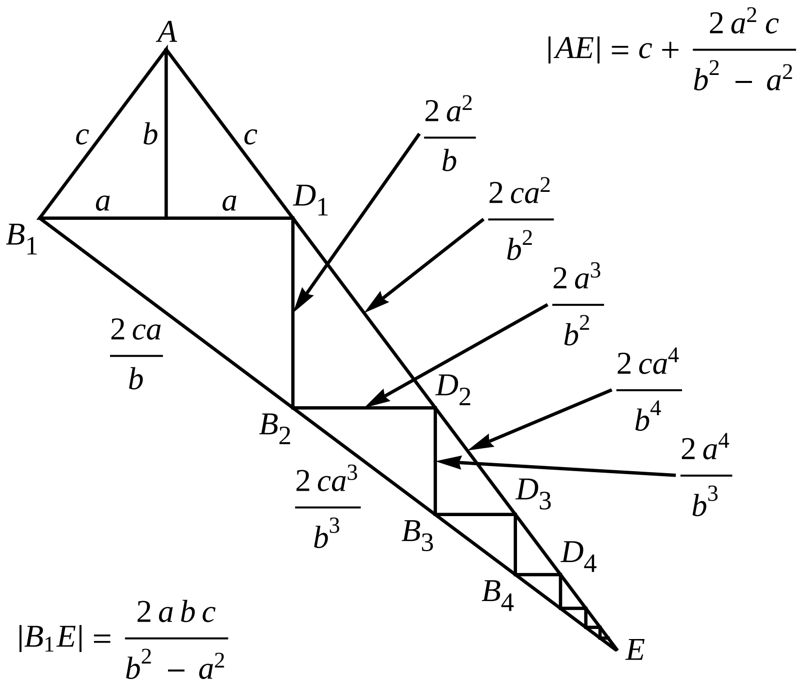
Hence

$$\begin{aligned} |B_1E| &= \frac{2ca}{b}\sum_{n=0}^{\infty}\left(\frac{a^2}{b^2}\right)^n = \frac{2abc}{b^2-a^2}, \\ |AE| &= c + 2c\sum_{n=1}^{\infty}\left(\frac{a^2}{b^2}\right)^n = c + \frac{2a^2c}{b^2-a^2}. \end{aligned}$$

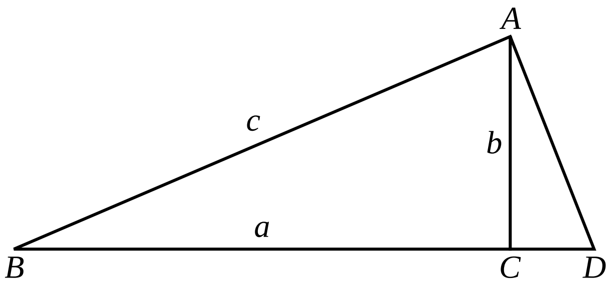
Observe that AB_1E is an arbitrary right-angled triangle. Its hypotenuse is AE, and

$$|AE|^2 - |B_1E|^2 = \left(c + \frac{2a^2c}{b^2-a^2}\right)^2 - \left(\frac{2abc}{b^2-a^2}\right)^2 = c^2 = |AB_1|^2.$$

QED



A simple proof, you will agree. However, **Graham Lovegrove** reminded the LSBU Maths Study Group of an even simpler proof, also based on triangles and the similarity thereof. Look at the next diagram ...



... in which angles ACB and BAD are 90°. Clearly,

$$|AD| = \frac{cb}{a}, \quad |CD| = \frac{b^2}{a}, \quad |BD| = \frac{c}{b}\,|AD| = \frac{c^2}{a},$$

and hence $c^2/a = a + b^2/a$.

Inspired by Sloman's article and the debates regarding the use or otherwise of trigonometry in Johnson & Jackson's argument, I thought I would have a go at proving Pythagoras' theorem in its most fundamental form. No diagram, no triangles, no trigonometry, and no funny stuff involving $\sqrt{-1}$.

Theorem 1 (Pythagoras) *We have*

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2 \;=\; 1.$$

Proof It is not too difficult to show that for positive integer n,

$$\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2 - 1.$$

is a polynomial in x of the form

$$P_n(x) \;=\; a_{2n+2}x^{2n+2} + a_{2n+4}x^{2n+4} + \cdots + a_{4n+2}x^{4n+2}, \tag{1}$$

where

$$|a_k| \;\le\; \frac{2^k}{k!}, \quad k = 2n+2,\ 2n+4,\ \ldots,\ 4n+2.$$

Hence for any x, $P_n(x) \to 0$ as $n \to \infty$. The theorem follows.

For completeness, we now address in detail the 'not too difficult' part of the proof. Let

$$\begin{aligned} C_n &= \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}, \\ S_n &= \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}, \end{aligned}$$

where n is a positive integer. Then

$$C_n^2 + S_n^2 \;=\; \sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{j+k}\left(\frac{x^{2j+2k}}{(2j)!(2k)!} + \frac{x^{2j+2k+2}}{(2j+1)!(2k+1)!}\right).$$

Put $r = k + j$:

$$C_n^2 + S_n^2 \;=\; \sum_{j=0}^{n}\sum_{r=j}^{n+j}(-1)^r\left(\frac{x^{2r}}{(2j)!(2r-2j)!} + \frac{x^{2r+2}}{(2j+1)!(2r-2j+1)!}\right).$$

Reverse the order of summation. Then r goes from 0 to $2n$, and we split its range into $[0,n]$ and $[n+1,2n]$. Observe that j goes from 0 to r when $r \le n$ and from $r-n$ to n when $r \ge n+1$. Thus we have

$$C_n^2 + S_n^2 \;=\; X_1 + X_2,$$

where

$$X_1 \;=\; \sum_{r=0}^{n}(-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=0}^{r} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=0}^{r} \binom{2r+2}{2j+1} \right),$$

$$X_2 \;=\; \sum_{r=n+1}^{2n}(-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=r-n}^{n} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=r-n}^{n} \binom{2r+2}{2j+1} \right).$$

Consider X_1. The two binomial sums can be evaluated:

$$\sum_{j=0}^{r} \binom{2r}{2j} \;=\; \begin{cases} 1 & \text{when } r = 0, \\ 2^{2r-1} & \text{otherwise,} \end{cases}$$

$$\sum_{j=0}^{r} \binom{2r+2}{2j+1} \;=\; 2^{2r+1}.$$

Therefore

$$\begin{aligned} X_1 \;&=\; 1 + \sum_{r=1}^{n}(-1)^r \frac{x^{2r}2^{2r-1}}{(2r)!} + \sum_{r=0}^{n}(-1)^r \frac{x^{2r+2}2^{2r+1}}{(2r+2)!} \\ &=\; 1 + x^2 + \sum_{r=1}^{n} \frac{(-1)^r}{2} \left(\frac{(2x)^{2r}}{(2r)!} + \frac{(2x)^{2r+2}}{(2r+2)!} \right). \end{aligned}$$

In this beautiful formula the last sum collapses into $-x^2$ (which cancels the x^2) and a term involving x^{2n+2}:

$$X_1 \;=\; 1 + \frac{(-1)^n}{2} \frac{(2x)^{2n+2}}{(2n+2)!}.$$

When we add X_2 to X_1 we do indeed get a polynomial of the form (1):

$$\begin{aligned} C_n^2 + S_n^2 - 1 \;&=\; \frac{(-1)^n}{2} \frac{(2x)^{2n+2}}{(2n+2)!} \\ &\quad + \sum_{r=n+1}^{2n}(-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=r-n}^{n} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=r-n}^{n} \binom{2r+2}{2j+1} \right). \end{aligned}$$

On gathering expressions involving the same power of x, we have

$$\begin{aligned} C_n^2 + S_n^2 - 1 \;&=\; (-1)^n \,\frac{x^{2n+2}}{(2n+2)!}\left(\frac{2^{2n+2}}{2} - \sum_{j=1}^{n}\binom{2n+2}{2j}\right) \\ &\quad + \frac{x^{4n+2}}{(4n+2)!}\binom{4n+2}{2n+1} \\ &\quad + \sum_{r=n+2}^{2n}\frac{(-1)^r x^{2r}}{(2r)!}\left(\sum_{j=r-n}^{n}\binom{2r}{2j} - \sum_{j=r-1-n}^{n}\binom{2r}{2j+1}\right). \end{aligned}$$

Next, observe that

$$\sum_{j=1}^{n}\binom{2n+2}{2j} \;=\; 2^{2n+1} - 2.$$

Therefore

$$\begin{aligned} C_n^2 + S_n^2 - 1 \;&=\; (-1)^n \,\frac{2\,x^{2n+2}}{(2n+2)!} + \frac{x^{4n+2}}{(4n+2)!}\binom{4n+2}{2n+1} \\ &\quad + \sum_{r=n+2}^{2n}\frac{(-1)^r x^{2r}}{(2r)!}\left(\sum_{j=r-n}^{n}\binom{2r}{2j} - \sum_{j=r-1-n}^{n}\binom{2r}{2j+1}\right). \end{aligned}$$

But $\binom{4n+2}{2n+1}$ is less than or equal to 2^{4n+2}. Moreover, the two inner sums involve distinct binomial coefficients of the form $\binom{2r}{s}$ and so when taken together they sum to something bounded by $\pm 2^{2r}$. Hence the coefficient of x^k in $C_n^2 + S_n^2 - 1$ is zero unless $k \in \{2n+2,\ 2n+4,\ \ldots,\ 4n+2\}$ in which case it is bounded by $\pm 2^k/k!$. □

If we want to avoid all those rather complicated and somewhat messy power series manipulations, there is an interesting alternative and much simpler argument. However, we have to assume some familiarity with high-school calculus, which might be considered an excessive demand for what should be a 'pure' proof of the Pythagorean theorem.

To emphasize the dependence of the sums on a variable, let us write

$$C(x) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, \qquad S(x) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

Lemma 1 *We have*

$$\frac{d}{dx}\Big(C^2(x) + S^2(x)\Big) \;=\; 0.$$

Proof On differentiating term by term we obtain

$$\begin{aligned} \frac{dC(x)}{dx} &= \sum_{k=1}^{\infty}(-1)^k \frac{2k\,x^{2k-1}}{(2k)!} \;=\; \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} \\ &= \sum_{j=0}^{\infty}(-1)^{j+1} \frac{x^{2j+1}}{(2j+1)!} \;=\; -\,S(x) \end{aligned}$$

and

$$\frac{dS(x)}{dx} \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!} \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \;=\; C(x).$$

Hence $\dfrac{d}{dx}\Big(C^2(x) + S^2(x)\Big) \;=\; -\,2C(x)S(x) + 2S(x)C(x) \;=\; 0.$ □

It is clear that $C^2(0) + S^2(0) = 1$. Therefore Theorem 1 follows from Lemma 1.

Pythagoras' theorem revisited again

Tony Forbes

My proof of the Pythagorean theorem in M500 **313** has been the subject of a certain amount criticism from various people to whom I have shown it. Leaving aside any questioning of the author's sanity, the main discussion was centered on my sledgehammer approach to the solution of a standard problem in high-school Euclidean geometry. Spread over three pages (albeit A5 ones) it is far too long-winded.

The power-series proof in M500 **313**, which involves computing (1) and then letting n tend to infinity, can be done much more easily by dealing directly with infinite sums. However, I claim that my proof is of some interest (at least to me). It avoids tortuous geometric reasoning involving such lofty concepts as 'angles', 'straight lines' and the relations between them. And since there isn't one, we don't have to worry about our diagram truly representing the general case.

I was quite keen to see exactly what happens when you compute the finite sum

$$\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2+\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2. \tag{1}$$

For instance, when $n=4$ you get

$$\frac{131681894400+72576x^{10}-30240x^{12}+2160x^{14}-63x^{16}+x^{18}}{131681894400},$$

and you can plainly see that this is nearly equal to 1 when $|x|$ is not too large. What has happened is that all the small positive powers of x have disappeared. For general n, as explained in M500 **313**, the non-constant part of the polynomial looks like this,

$$a_{2n+2}x^{2n+2}+a_{2n+4}x^{2n+4}+\cdots+a_{4n+2}x^{4n+2},$$

with the a_k bounded by $\pm 2^k/k!$. Thus we are assured of rapid convergence to zero for any x. Just make n go to ∞.

For the critics, I offer the suggested alternative proof of the theorem. Like the one in M500 **313**, there is no mention of triangles or trigonometry. We achieve considerable simplicity because we avoid the complicated details associated with manipulating finite sums. However, the simplicity is an illusion. We are not bothering to answer thorny questions concerning the convergence of any infinite power series that appears in our argument.

Theorem 1 (Pythagoras) *We have*

$$\left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)^2 \;=\; 1.$$

Proof Let

$$C(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}, \qquad S(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Then

$$C^2(x) \;=\; \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k} \frac{x^{2j+2k}}{(2j)!(2k)!}.$$

Put $r = k + j$:

$$C^2(x) \;=\; \sum_{r=0}^{\infty}\sum_{j=0}^{r}(-1)^r \frac{x^{2r}}{(2j)!(2r-2j)!} \;=\; \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r}}{(2r)!}\sum_{j=0}^{r}\binom{2r}{2j}.$$

The binomial sum can be evaluated: it is 1 when $r = 0$ and 2^{2r-1} when $r \geq 1$. Hence

$$C^2(x) \;=\; 1 + \frac{1}{2}\sum_{r=1}^{\infty}(-1)^r \frac{(2x)^{2r}}{(2r)!} \;=\; \frac{1 + C(2x)}{2}. \tag{2}$$

Similarly,

$$\begin{aligned} S^2(x) \;&=\; \sum_{r=0}^{\infty}\sum_{j=0}^{r}(-1)^r \frac{x^{2r+2}}{(2j+1)!(2r-2j+1)!} \\ &=\; \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r+2}}{(2r+2)!}\sum_{j=0}^{r}\binom{2r+2}{2j+1} \\ &=\; \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r \frac{(2x)^{2r+2}}{(2r+2)!} \;=\; \frac{1 - C(2x)}{2} \end{aligned}$$

since the binomial sum is 2^{2r+1} for $r \geq 0$. Combining with (2), we have

$$C^2(x) + S^2(x) \;=\; 1. \qquad \square$$

If we permit the use of $i = \sqrt{-1}$, the previous argument can be expressed even more succinctly. Define

$$E(x) \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and observe that

$$\begin{aligned} E(x)E(y) \;&=\; \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j}{j!}\frac{y^k}{k!} \;=\; \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{j=0}^{r} x^j y^{r-j} \binom{r}{j} \\ &=\; \sum_{r=0}^{\infty} \frac{1}{r!}\,(x+y)^r \;=\; E(x+y). \end{aligned} \tag{3}$$

Also one can easily verify the familiar identities

$$E(ix) \;=\; C(x) + iS(x), \qquad E(-ix) \;=\; C(x) - iS(x) \tag{4}$$

by splitting the sum for $E(\pm ix)$ into a part that does not explicitly involve i and a part that does.

Now the proof of Theorem 1 is extremely straightforward:

$$\begin{aligned} C^2(x) + S^2(x) \;&=\; \big(C(x) + iS(x)\big)\big(C(x) - iS(x)\big) \\ &=\; E(ix)E(-ix) \;=\; E(0) \;=\; 1. \end{aligned}$$

By looking at the power series expressions, we see immediately that $E(0) = C(0) = 1$, $S(0) = 0$. And I believe that $E(1) \approx 2.71828$ is an important mathematical constant.

By computing its power series to sufficiently many terms we can verify that $S(x)$ has a zero in the vicinity of $22/7$. So let us define a number, which we shall call π, by

$$S(\pi) \;=\; 0, \qquad 3.14 \;\le\; \pi \;\le\; 3.15. \tag{5}$$

This number has some interesting properties. Using Pythagoras' theorem we obtain $C(\pi) = \pm 1$, and by comparing with the value obtained from directly computing the power series, $C(\pi) \approx -1.0$, it is clear that we must select the negative sign; thus $C(\pi) = -1$ exactly. Then, by making use of (3) and (4), we have

$$E(i\pi) \;=\; -\,1, \qquad E(2i\pi) \;=\; 1,$$

$$\begin{aligned} E(x+2i\pi) &= E(x)E(2i\pi) \;=\; E(x), \\ C(x+2\pi) &= C(x), \qquad S(x+2\pi) \;=\; S(x). \end{aligned}$$

Thus we have shown that $C(x)$ and $S(x)$ are periodic with period 2π. Moreover, it is not difficult to prove that whenever j is an integer

$$\begin{aligned} S(j\pi) &= C\left(j\pi + \frac{\pi}{2}\right) = 0, \\ C(2j\pi) &= S\left(2j\pi + \frac{\pi}{2}\right) = 1, \\ C(2j\pi+\pi) &= S\left(2j\pi - \frac{\pi}{2}\right) = -1, \end{aligned}$$

and, again with the help of (3) and Pythagoras' theorem, we can obtain values at other rational multiples of π, such as

$$S\left(\frac{\pi}{4}\right) = C\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

$$S\left(\frac{\pi}{3}\right) = C\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \qquad S\left(\frac{\pi}{6}\right) = C\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

$$S\left(\frac{\pi}{5}\right) = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \quad C\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}.$$

The proofs are straightforward and left to the reader.

Finally, let us define yet another function by a power series:

$$\begin{aligned} D(x) &= \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{x^{2k+1}}{2k+1} \\ &= x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816} - \frac{21x^{13}}{13312} - \frac{11x^{15}}{10240} - \dots, \end{aligned}$$

valid for $-1 \le x \le 1$ (the coefficient of x^{2k+1} is $O(k^{-5/2})$; see Problem 314.3 on page 11). By computing the power series to sufficient accuracy, you can verify that $D(1) \approx \pi/4$. Moreover, you probably recognise $D(x)$ as $\int_0^x \sqrt{1-u^2}\,du$, and so possibly one can argue that $4D(1)$ is the area of a unit circle—whatever that might mean. So we offer an interesting problem for you to solve.

Prove that $4D(1) = \pi$, where π is defined by (5).

Problem 314.3 – Binomial coefficient

Tony Forbes

Show that for large n,

$$\binom{1/2}{n} \sim \frac{(-1)^{n+1}}{2\sqrt{\pi}\, n^{3/2}}.$$

The expression on the left is interpreted as

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!},$$

i.e. half choose n, the number of ways to select n objects from half an object.