Towers: Hanoi, Saigon and beyond

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For integers $p \geq 2$ and $n \geq 1$, we define the function $T_p(n)$ by

$$T_2(n) = \infty \quad \text{for } n \geq 2, \qquad T_p(1) = 1 \quad \text{for } p \geq 2, \tag{1}$$

$$T_p(n) = \min_{1 \leq k \leq n-1} \left\{ 2T_p(k) + T_{p-1}(n-k) \right\}. \tag{2}$$

So $T_2(n)$ is completely defined by (1) and it is not difficult to compute $T_3(n)$. Observe that when $p = 3$ the expression inside the curly brackets on the right of (2) is finite only when $k = n - 1$. Hence

$$T_3(n) = 2T_3(n-1) + T_2(1) = 2T_3(n-1) + 1,$$

which when combined with (1) gives $T_3(n) = 2^n - 1$. For $p \geq 4$ things are not so easy, at least for large n, and I don't immediately see any alternative to evaluating a large number of minima over k. The results of some computations are tabulated on page 2. To save space I did not include a bulky column for $T_3(n)$.

As you can see from the table, $T_p(n)$ seems to become constant when p gets large, and indeed one might conjecture that

$$T_n(n) = 2n + 1 \quad \text{and} \quad T_p(n) = 2n - 1 \text{ whenever } p > n. \tag{3}$$

At this point it is appropriate to describe a generalization of a familiar puzzle involving the transfer of a tower of discs.

There are p vertical pegs lined up in a row and n discs of distinct radii. The discs have holes in their centres so that they can be threaded on to the pegs. Initially, all n discs are placed on the left-hand peg in descending order of size to form a conical tower, as shown on the left in the picture on page 3. The object of the game is to transfer the entire tower to the right-hand peg. Discs are moved from peg to peg, one at a time according to the following rules:

(i) only a disc at the top of a tower may be moved;

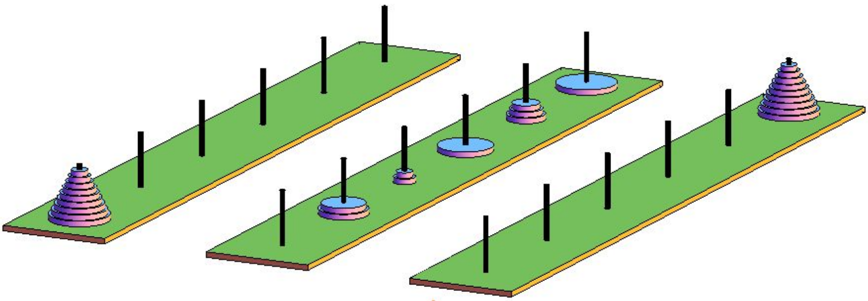
(ii) you must never put a disc on top of a smaller disc.

How many moves are needed?

If $p = 2$, there is not a lot you can do. You are stuck unless $n = 1$ and then it takes only one move to transfer the single disc.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
4	9	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
5	13	11	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
6	17	15	13	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
7	25	19	17	15	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	33	23	21	19	17	15	15	15	15	15	15	15	15	15	15	15	15	15
9	41	27	25	23	21	19	17	17	17	17	17	17	17	17	17	17	17	17
10	49	31	29	27	25	23	21	19	19	19	19	19	19	19	19	19	19	19
11	65	39	33	31	29	27	25	23	21	21	21	21	21	21	21	21	21	21
12	81	47	37	35	33	31	29	27	25	23	23	23	23	23	23	23	23	23
13	97	55	41	39	37	35	33	31	29	27	25	25	25	25	25	25	25	25
14	113	63	45	43	41	39	37	35	33	31	29	27	27	27	27	27	27	27
15	129	71	49	47	45	43	41	39	37	35	33	31	29	29	29	29	29	29
16	161	79	57	51	49	47	45	43	41	39	37	35	33	31	31	31	31	31
17	193	87	65	55	53	51	49	47	45	43	41	39	37	35	33	33	33	33
18	225	95	73	59	57	55	53	51	49	47	45	43	41	39	37	35	35	35
19	257	103	81	63	61	59	57	55	53	51	49	47	45	43	41	39	37	37
20	289	111	89	67	65	63	61	59	57	55	53	51	49	47	45	43	41	39
21	321	127	97	71	69	67	65	63	61	59	57	55	53	51	49	47	45	43
22	385	143	105	79	73	71	69	67	65	63	61	59	57	55	53	51	49	47
23	449	159	113	87	77	75	73	71	69	67	65	63	61	59	57	55	53	51
24	513	175	121	95	81	79	77	75	73	71	69	67	65	63	61	59	57	55
25	577	191	129	103	85	83	81	79	77	75	73	71	69	67	65	63	61	59
26	641	207	137	111	89	87	85	83	81	79	77	75	73	71	69	67	65	63
27	705	223	145	119	93	91	89	87	85	83	81	79	77	75	73	71	69	67
28	769	239	153	127	97	95	93	91	89	87	85	83	81	79	77	75	73	71
29	897	255	161	135	105	99	97	95	93	91	89	87	85	83	81	79	77	75
30	1025	271	169	143	113	103	101	99	97	95	93	91	89	87	85	83	81	79
31	1153	287	177	151	121	107	105	103	101	99	97	95	93	91	89	87	85	83
32	1281	303	185	159	129	111	109	107	105	103	101	99	97	95	93	91	89	87
33	1409	319	193	167	137	115	113	111	109	107	105	103	101	99	97	95	93	91
34	1537	335	201	175	145	119	117	115	113	111	109	107	105	103	101	99	97	95
35	1665	351	209	183	153	123	121	119	117	115	113	111	109	107	105	103	101	99
36	1793	383	225	191	161	127	125	123	121	119	117	115	113	111	109	107	105	103
37	2049	415	241	199	169	135	129	127	125	123	121	119	117	115	113	111	109	107
38	2305	447	257	207	177	143	133	131	129	127	125	123	121	119	117	115	113	111
39	2561	479	273	215	185	151	137	135	133	131	129	127	125	123	121	119	117	115
40	2817	511	289	223	193	159	141	139	137	135	133	131	129	127	125	123	121	119
41	3073	543	305	231	201	167	145	143	141	139	137	135	133	131	129	127	125	123
42	3329	575	321	239	209	175	149	147	145	143	141	139	137	135	133	131	129	127
43	3585	607	337	247	217	183	153	151	149	147	145	143	141	139	137	135	133	131
44	3841	639	353	255	225	191	157	155	153	151	149	147	145	143	141	139	137	135
45	4097	671	369	263	233	199	161	159	157	155	153	151	149	147	145	143	141	139
46	4609	703	385	271	241	207	169	163	161	159	157	155	153	151	149	147	145	143
47	5121	735	401	279	249	215	177	167	165	163	161	159	157	155	153	151	149	147
48	5633	767	417	287	257	223	185	171	169	167	165	163	161	159	157	155	153	151
49	6145	799	433	295	265	231	193	175	173	171	169	167	165	163	161	159	157	155
50	6657	831	449	303	273	239	201	179	177	175	173	171	169	167	165	163	161	159

The case $p = 3$ is *The Tower of Hanoi*, and the solution, $H(n)$, is well known. First transfer the top $n-1$ discs from peg 1 to peg 2 ($H(n-1)$ moves) and then transfer them to peg 3 (another $H(n-1)$ moves) after placing the largest disc on to peg 3 (1 move). Therefore $H(n) = 2H(n-1) + 1$, which when combined with $H(1) = 1$ gives $H(n) = 2^n - 1$. The puzzle is solved and $H(n) = T_3(n)$.



We discussed $p = 4$ in M500 **163**, where it was called *The Tower of Saigon*. There we described the following strategy. (i) Transfer the top k discs to a spare peg using *Tower of Saigon* moves; (ii) put the remaining $n-k$ discs on the destination peg using *Tower of Hanoi* moves (because the peg that now contains the first k discs is not available); (iii) transfer the k smallest discs to the destination peg (*Tower of Saigon* moves). We choose k to minimize the number of moves, $S(n)$. Thus $S(1) = 1$ and

$$S(n) \;=\; \min_{1\le k\le n-1} \Big\{ 2S(k) + H(n-k) \Big\};$$

hence $S(n) = T_4(n)$. This is the *Frame–Stewart algorithm* [1]. However, it is has never been proved that $T_4(n)$ represents the best solution. The Tower of Saigon problem is therefore still open—although it is known that $T_4(n)$ is optimal for $n \le 30$.

From the table we see that the differences $\Delta T_4(n) = T_4(n) - T_4(n-1)$ look like this:

$$\Delta T_4(n) \;=\; 2, 2, 4, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 32, 32, 32, 32, 32, 32, 64, \ldots.$$

In [2] it is proved that the pattern continues indefinitely, and the closed formula (also in [2]) was obtained by Peter Fletcher in M500 **166**:

$$T_4(n) \;=\; 2^r \left(n - \frac{r(r-1)}{2} - 1\right) + 1, \quad \text{where } r \;=\; \left\lceil \frac{\sqrt{8n+1} - 3}{2} \right\rceil. \qquad (4)$$

Applying the same strategy to larger puzzles provides the best known solutions: $T_p(n)$ for p pegs, $p \geq 5$. But since they all involve the unproven $T_4(n)$ for four pegs we cannot be sure that they are optimal for large n. It occurred to me that the procedure for 5 pegs, say, might be improved by having two parameters j and k. We transfer the top k discs to a spare peg using 5-peg moves, transfer the next j discs to another spare peg using *Saigon* moves and the final $n-j-k$ discs to the destination peg with *Hanoi* moves. The new solution satisfies

$$\overline{T}_5(n) \;=\; \min_{1\leq k\leq n-1,\ 1\leq j\leq n-k} \left\{ 2\overline{T}_5(k) + 2S(j) + H(n-k-j) \right\};$$

but $2S(j) + H(n-k-j) \geq S(n-k)$ by (2) and we have gained nothing.

For $p \geq 5$, the differences appear to exhibit the same kind of pattern involving powers of 2. Thus, writing 2, 2, 2 as $2\!:\!3$, for example, we have

$$\begin{aligned}
\Delta T_5(n) &= 2:3,\ 4:6,\ 8:10,\ 16:15,\ 32:21,\ 64:28,\ \ldots, \\
\Delta T_6(n) &= 2:4,\ 4:10,\ 8:20,\ 16:35,\ 32:56,\ 64:84,\ \ldots, \\
&\ldots, \\
\Delta T_p(n) &= 2:p-2,\ 4:\binom{p-1}{2},\ 8:\binom{p}{3},\ 16:\binom{p+1}{4},\ \ldots
\end{aligned}$$

and in general 2^r occurs $\binom{p+r-3}{r} = \binom{p+r-3}{p-3}$ times in the sequence. This also holds for $p=3$ since $T_3(n) - T_3(n-1) = 2^{n-1}$ and the binomial coefficient is always 1.

Unfortunately I have no proof that when $p \geq 5$ the differences $\Delta T_p(n)$ actually have the stated properties. A gap which the reader might like to fill. In view of this ignorance, let us define a new function $U_p(n)$.

Let $U_2(n) = T_2(n)$. For $p \geq 3$, let $U_p(1) = 1$ and when $n \geq 2$ we assume that the differences $\Delta U_p(n) = U_p(n) - U_p(n-1)$ have the properties stated above. Thus for $p \geq 3$ and $n \geq 2$,

$$\Delta U_p(n) \;=\; 2:\binom{p-2}{1},\ 4:\binom{p-1}{2},\ 8:\binom{p}{3},\ \ldots,\ 2^r:\binom{p+r-3}{p-3},\ \ldots.$$

Then we know that $U_3(n) = T_3(n)$ and $U_4(n) = T_4(n)$, and in the absence of a proof we conjecture that $U_p(n) = T_p(n)$ when $p \geq 5$. To obtain a closed formula for $U_p(n)$ we split n into two parts, m and $n-m$, where

$$m \;=\; 1 + \sum_{i=1}^{r-1} \binom{p-3+i}{p-3} \;=\; \binom{p-3+r}{p-2},$$

and r is the positive integer satisfying

$$\binom{p-3+r}{p-2} \;<\; n \;\leq\; \binom{p-2+r}{p-2}. \tag{5}$$

Then by starting with 1 and summing the differences we have

$$U_p(n) \;=\; 1+\sum_{j=1}^{r-1} 2^j \binom{p-3+j}{p-3} + 2^r \left(n - \binom{p-3+r}{p-2}\right), \quad p \geq 3,\; n \geq 2,$$

where r satisfies (5). The main advantage of course is that we avoid all those computations to determine the minima in (2).

To see how it works, let $p = 3$. The inequalities (5) defining r reduce to $r = n-1$ and so

$$U_3(n) \;=\; 1+\sum_{j=1}^{n-2} 2^j + 2^{n-1} \;=\; 2^n - 1 \;=\; T_3(n) \;=\; H(n).$$

When $p = 4$ it is a little more complicated. Now the condition (5) becomes $(r+1)r < 2n \leq (r+2)(r+1)$, and to get an explicit value for r we solve $2n = (r+2)(r+1)$ and round up. So $r = \lceil(\sqrt{8n+1}-3)/2\rceil$ and

$$U_4(n) \;=\; 1+\sum_{j=1}^{r-1} 2^j (j+1) + 2^r \left(n - \frac{r(r+1)}{2}\right).$$

Since the sum evaluates to $2^r(r-1)$, we obtain the same formula as in (4):

$$U_4(n) \;=\; 2^r \left(n - \frac{r(r-1)}{2} - 1\right) + 1, \quad \text{where } r \;=\; \left\lceil \frac{\sqrt{8n+1}-3}{2} \right\rceil.$$

By a similar kind of argument one can obtain formulae for further values of p:

$$\begin{aligned}
U_5(n) \;&=\; 2^r \left(n - \frac{(r-1)(r^2+r+6)}{6}\right) - 1,\\
U_6(n) \;&=\; 2^r \left(n - \frac{(r-1)r(r^2+3r+14)}{24} - 1\right) + 1,\\
U_7(n) \;&=\; 2^r \left(n - \frac{(r-1)(r^4+6r^3+31r^2+26r+120)}{120}\right) - 1,\\
U_8(n) \;&=\; 2^r \left(n - \frac{(r-1)r(r^4+10r^3+65r^2+140r+444)}{720} - 1\right) + 1
\end{aligned}$$

and so on, with r defined by

$$r \;=\; \lceil x \rceil, \quad \text{where} \quad \binom{p-2+x}{p-2} = n \quad \text{and} \quad x \geq 0.$$

However, apart from $p = 3$, 4, 5, 6, 8 and 10, I do not know of any case where is possible to get a closed formula for r in terms of n. Cases $p = 8$ and 10 involve polynomials with messy solutions but for $p = 6$ the quartic is quite easy to solve:

$$\binom{x+4}{4} = n \;\text{ and }\; x \geq 0 \quad \Rightarrow \quad x = \frac{\sqrt{5+4\sqrt{24n+1}}-5}{2}.$$

So for computing $U_6(n)$ we set $r = \left\lceil \frac{1}{2}\left(\sqrt{5+4\sqrt{24n+1}}-5\right)\right\rceil$. And by solving $(x+3)(x+2)(x+1) = 6n$ we obtain a slightly more complicated r for computing $U_5(n)$:

$$r \;=\; \left\lceil \sqrt[3]{3n+\sqrt{9n^2-\frac{1}{27}}} + \sqrt[3]{3n-\sqrt{9n^2-\frac{1}{27}}} - 2 \right\rceil.$$

Finally, we can offer a simple explanation of (3) in terms of the puzzle. If $p > n$, there are enough spare pegs to store $n-1$ discs, one per peg. So we do $n-1$ moves to get rid of the top $n-1$ discs, one more to transfer the nth disc and another $n-1$ to pile the rest on top of it; $2n-1$ altogether. This corresponds to making the choice $k = 1$ at each stage of the recursion formula (2):

$$\begin{aligned} T_p(n) \;&=\; 2T_p(1) + T_{p-1}(n-1) \;=\; 2 + T_{p-1}(n-1) \;=\; 4 + T_{p-2}(n-2) \\ &=\; \ldots \;=\; 2(n-2) + T_{p-n+2}(2) \;=\; 2(n-1) + T_{p-n+1}(1) \;=\; 2n-1 \end{aligned}$$

since $T_{p-n+1}(1) = 1$. Similarly but with a bit more work we can derive

$$T_p(n) \;=\; 4n - 2p + 1 \quad \text{for} \;\; p \leq n \leq \frac{p(p-1)}{2}.$$

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[1] B. M. Stewart and J. S. Frame, Solution to advanced problem 3819. *American Mathematical Monthly* **48** 3 (March 1941), 2169.

[2] Paul K. Stockmeyer, Variations on the Four-Post Tower of Hanoi Puzzle, *Congressus Numerantium* **102** (1994), 3–12.