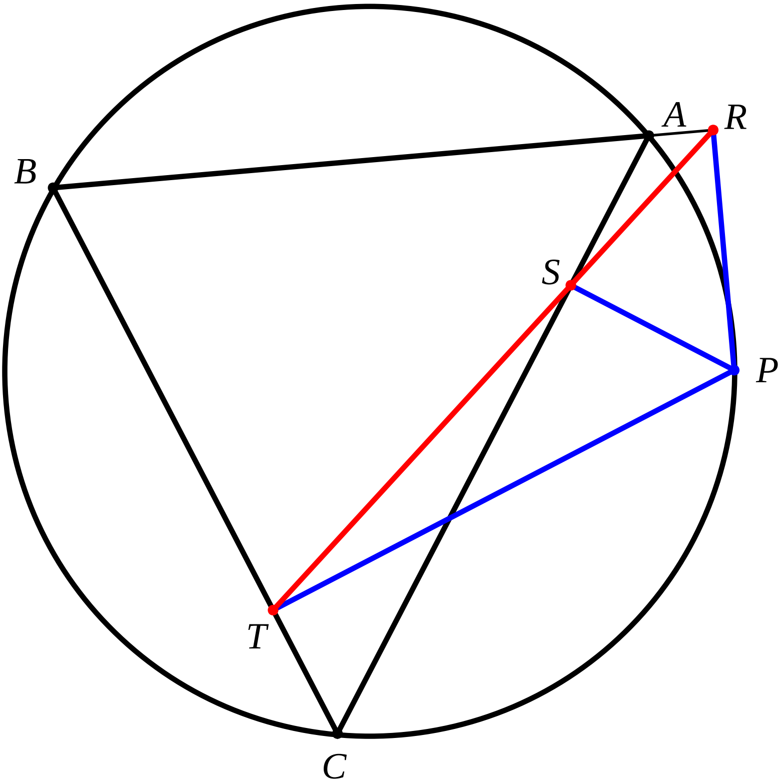
The Wallace–Simson theorem

Tony Forbes

Let

$$A = (\cos a, \sin a), \quad B = (\cos b, \sin b), \quad C = (\cos c, \sin c), \quad P = (1, 0),$$

and suppose R is on AB such that PR is perpendicular to AB, S is on AC such that PS is perpendicular to AC, T is on BC such that PT is perpendicular to BC. The Wallace–Simson theorem says that R, S and T are collinear.



There are various clever ways of establishing this result using geometry or complex numbers or perhaps some other method; see *Wikipedia*, for example. Here I offer a proof that does not rely on human ingenuity: simply compute the coordinates of R, S and T by brute force.

Consider R. This point must satisfy three equalities,

$$\begin{aligned} A + r(B - A) &= R, \\ (R - A) \cdot (R - P) &= 0, \\ (R - B) \cdot (R - P) &= 0 \end{aligned} \tag{1}$$

for some value of r. The first expresses the collinearity of A, B and R. The other two follow from the orthogonality relations $RA \perp RP$ and $RB \perp RP$.

The equations (1) are solved by the traditional method. We write down the solution,

$$\begin{aligned} R &= \left(\frac{1}{2} \big(\cos a + \cos b - \cos(a+b) + 1 \big),\ \frac{1}{2} \big(\sin a + \sin b - \sin(a+b) \big) \right), \\ r &= \frac{\sin(a/2)\ \cos(b/2)}{\sin((a-b)/2)}, \end{aligned}$$

and invite the reader to verify that it satisfies (1). For the other two points, we get similar formulæ:

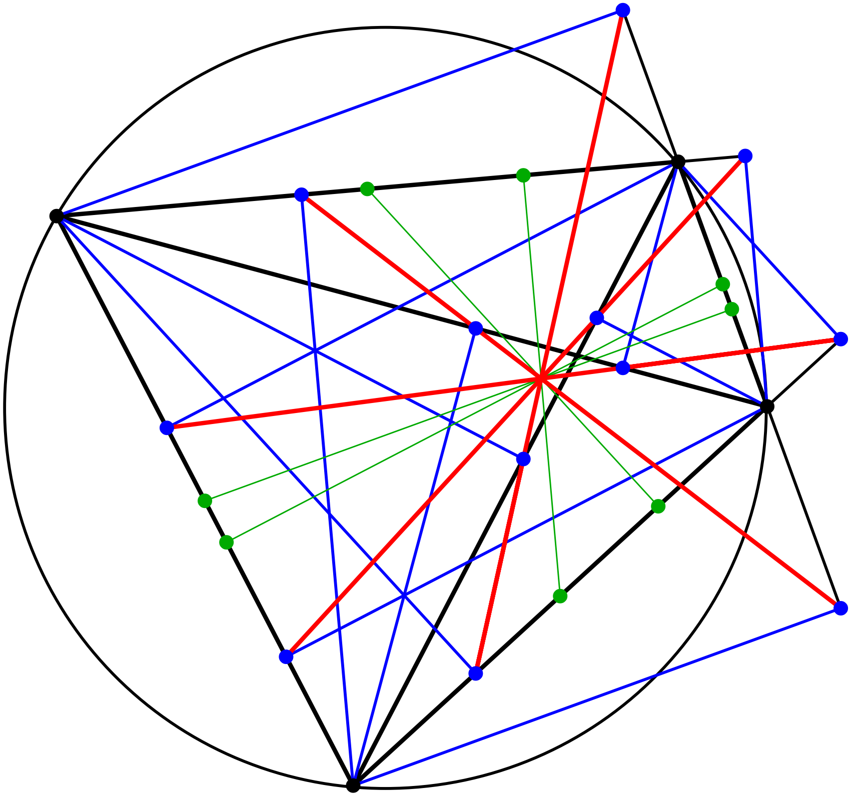
$$\begin{aligned} S &= \left(\frac{1}{2} \big(\cos a + \cos c - \cos(a+c) + 1 \big),\ \frac{1}{2} \big(\sin a + \sin c - \sin(a+c) \big) \right), \\ T &= \left(\frac{1}{2} \big(\cos b + \cos c - \cos(b+c) + 1 \big),\ \frac{1}{2} \big(\sin b + \sin c - \sin(b+c) \big) \right). \end{aligned}$$

Finally, observe that if

$$u = \sin(a/2)\ \sin((b-c)/2), \quad \text{and} \quad v = \sin(b/2)\ \sin((a-c)/2),$$

then we have $u(T - R) = v(S - R)$; see Problem xxxxx. Hence R, S and T are collinear.

In the interests of symmetry we should regard $\mathcal{Q} = \{A, B, C, P\}$ as a set of four typical points on the circle. Then it makes good sense to construct the Wallace–Simson line for each triangle formed from three points of the cyclic quadrilateral $\mathcal{Q}$ with respect to the 4th point. In the diagram on the next page the four Wallace–Simson lines are shown in red and, as you can see, they meet at a common point, W. Moreover, for each side of $\mathcal{Q}$, one can construct a line that is perpendicular to this side and passes through the mid point of the opposite side (green in the diagram). These four lines also meet at W, [R. F. Cyster, The Simson lines of a cyclic quadrilateral, *Math. Gazette* **25**, (1941), 56–58, `https://doi.org/10.2307/3606490`].



Problem xxxxx – The Wallace–Simson line

Tony Forbes

Let

$$R(a,b) \;=\; \big(\cos a+\cos b-\cos(a+b)+1,\;\; \sin a+\sin b-\sin(a+b)\big)$$

and

$$u(a,b) \;=\; \sin(a/2)\;\sin((b-c)/2).$$

Show that

$$u(a,b)(R(b,c)-R(a,b)) \;=\; u(b,a)(R(a,c)-R(a,b)).$$