Bernoulli Polynomials
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Bernoulli Polynomials

The *Bernoulli polynomials* $B_n(x)$ are defined by

$$B_0(x) \;=\; 1,$$

$$B_n'(x) \;=\; nB_{n-1}(x) \quad \text{and} \quad \int_0^1 B_n(x)\,dx \;=\; 0, \quad n \ge 1. \tag{1}$$

Thus

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$

$$B_5(x) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}, \quad B_6(x) = x^6 - 3x^5 + \frac{5x^4}{2} - \frac{x^2}{2} + \frac{1}{42},$$

$$B_7(x) = x^7 - \frac{7x^6}{2} + \frac{7x^5}{2} - \frac{7x^3}{6} + \frac{x}{6}, \quad B_8(x) = x^8 - 4x^7 + \frac{14x^6}{3} - \frac{7x^4}{3} + \frac{2x^2}{3} - \frac{1}{30},$$

$$B_9(x) = x^9 - \frac{9x^8}{2} + 6x^7 - \frac{21x^5}{5} + 2x^3 - \frac{3x}{10}, \quad B_{10}(x) = x^{10} - 5x^9 + \frac{15x^8}{2} - 7x^6 + 5x^4 - \frac{3x^2}{2} + \frac{5}{66},$$

$$B_{11}(x) = x^{11} - \frac{11x^{10}}{2} + \frac{55x^9}{6} - 11x^7 + 11x^5 - \frac{11x^3}{2} + \frac{5x}{6},$$

$$B_{12}(x) = x^{12} - 6x^{11} + 11x^{10} - \frac{33x^8}{2} + 22x^6 - \frac{33x^4}{2} + 5x^2 - \frac{691}{2730}, \quad \ldots.$$

The constant term gives the nth Bernoulli number, $B_n = B_n(0)$,

$$B_0 = 1,\; B_1 = -\frac{1}{2},\; B_2 = \frac{1}{6},\; B_3 = 0,\; B_4 = -\frac{1}{30},\; B_5 = 0,\; B_6 = \frac{1}{42},\; B_7 = 0,\; B_8 = -\frac{1}{30},$$

$$B_9 = 0,\; B_{10} = \frac{5}{66},\; B_{11} = 0,\; B_{12} = -\frac{691}{2730},\; B_{13} = 0,\; B_{14} = \frac{7}{6},\; B_{15} = 0,\; B_{16} = -\frac{3617}{510},$$

$$B_{17} = 0, \quad B_{18} = \frac{43867}{798}, \quad B_{19} = 0, \quad B_{20} = -\frac{174611}{330}, \quad \ldots.$$

There is some merit in defining what one might call the *alternative Bernoulli polynomials*:

$$\widehat{B}_0(x) \;=\; 1,$$

$$\widehat{B}_n'(x) \;=\; n\widehat{B}_{n-1}(x) \quad \text{and} \quad \int_0^1 \widehat{B}_n(x)\,dx \;=\; 1, \quad n \ge 1.$$

Then $\widehat{B}_n(x) = B_n(x) + nx^{n-1} = (-1)^n B_n(-x)$. The corresponding *alternative Bernoulli numbers* are $\widehat{B}_n = \widehat{B}_n(0) = B_n(1)$. The only difference occurs when $n = 1$:

$$\widehat{B}_1 \;=\; \frac{1}{2} \;=\; -B_1, \qquad \widehat{B}_n \;=\; B_n, \quad n = 0,\ 2,\ 3,\ 4,\ \ldots.$$

We won't investigate the alternative polynomials, but watch out for $\widehat{B}_n$ in what follows.

The graphs show that $B_n(1-x) = (-1)^n B_n(x)$. From the definition, the coefficient of x is $B_n'(0) = nB_{n-1}$. More generally, the coefficient of x^m in $B_n(x)$ is $\binom{n}{m} B_{n-m}$; hence

$$B_n(x) \;=\; \sum_{m=0}^{n} \binom{n}{m} B_{n-m} x^m.$$

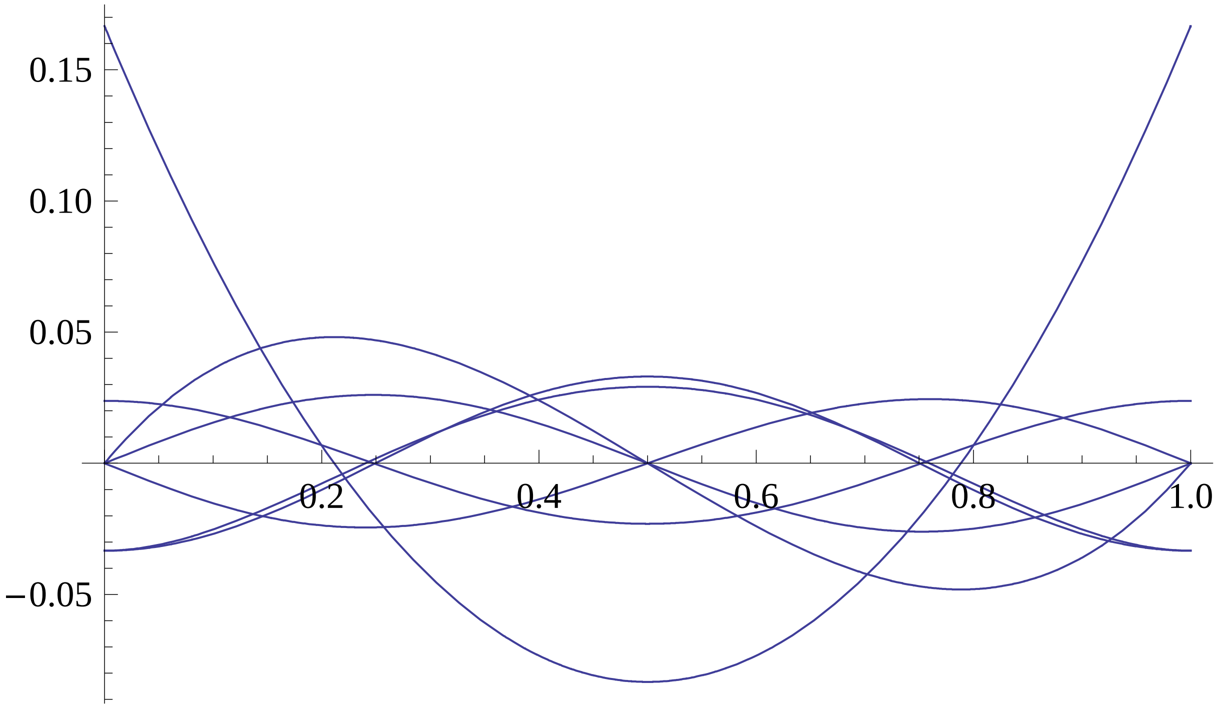
Moreover, for $x \in [0,1]$ we have these approximations for large n:

$$B_n(x) \;\approx\; B_n \cos 2\pi x, \quad n \text{ even}, \qquad B_n(x) \;\approx\; B_n(1/4) \sin 2\pi x, \quad n \text{ odd},$$

and

$$\int_0^1 B_n(x) B_m(x)\, dx \;=\; 0 \qquad \text{when } n+m \text{ is odd.}$$

Exercise for reader: compute $\int_0^1 B_n(x) B_m(x)\, dx$ when $n+m$ is even.



Bernoulli polynomials $B_n(x)$ for $n = 2, 3, \ldots, 8$

The generating functions for the Bernoulli polynomials and the Bernoulli numbers are respectively

$$\frac{te^{xt}}{e^t - 1} \;=\; \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad \text{and} \qquad \frac{t}{e^t - 1} \;=\; \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2}$$

These could be used as definitions; then the left-hand equality in (1) is recovered by differentiating the left-hand equality in (2) with respect to x:

$$\frac{t^2 e^{xt}}{e^x - 1} = \sum_{n=1}^{\infty} B_n'(x) \frac{t^n}{n!} \;\Rightarrow\; \frac{te^{xt}}{e^x - 1} = \sum_{n=1}^{\infty} B_n'(x) \frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{B_{n+1}'(x)}{n+1} \cdot \frac{t^n}{n!}.$$

From the right-hand equality in (2) we obtain two facts: (i) $B_{2n+1} = 0$ for $n \geq 1$ since the function $t/(e^t - 1) + t/2$ is even, and (ii) multiplying the series by the Taylor expansion of $(e^t - 1)/t$ gives

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k \;=\; 0^n.$$

This last equality can be used to compute the B_n recursively; see Lovelace [3], for example.

Stirling numbers of the second kind

The *Stirling number of the second kind* $S(n,k)$ is the number of ways to partition a set of n objects into k non-empty subsets:

$$S(n,k) \;=\; \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \;=\; \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n,$$

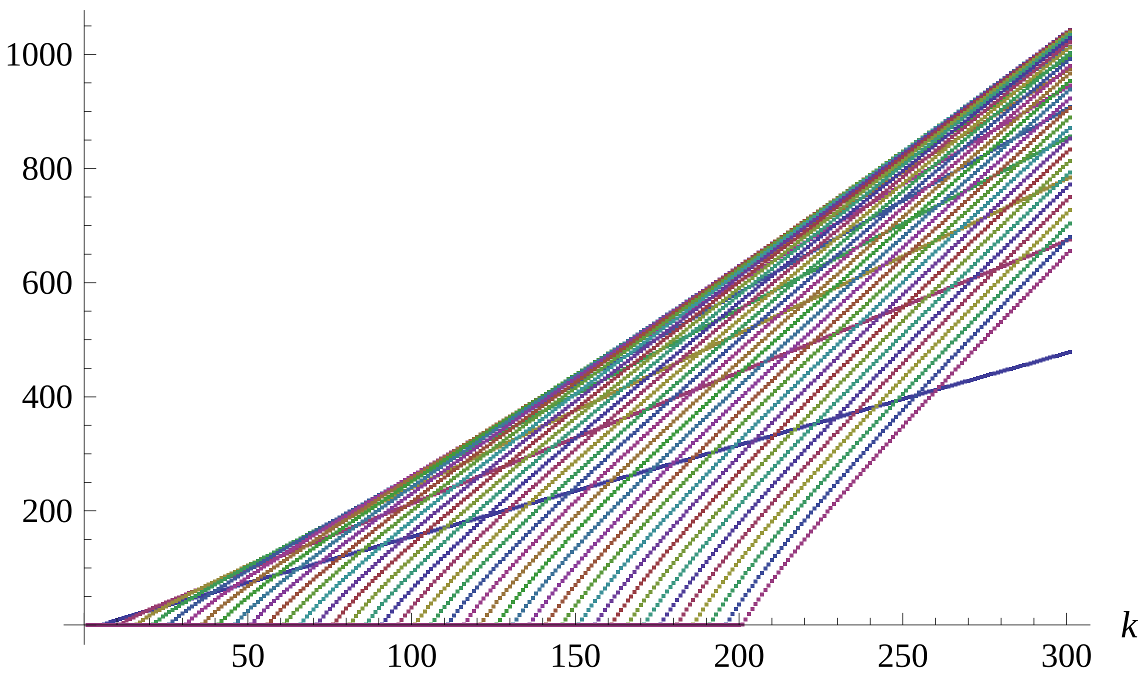
or they can be defined recursively

$$S(0,0) \;=\; 1, \qquad S(n,0) \;=\; 0 \text{ for } n>0, \qquad S(0,k) \;=\; 0 \text{ for } k>0,$$

$$S(n+1,k) \;=\; kS(n,k)+S(n,k-1) \text{ for } k>0.$$

						k					
n	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0
3	0	1	3	1	0	0	0	0	0	0	0
4	0	1	7	6	1	0	0	0	0	0	0
5	0	1	15	25	10	1	0	0	0	0	0
6	0	1	31	90	65	15	1	0	0	0	0
7	0	1	63	301	350	140	21	1	0	0	0
8	0	1	127	966	1701	1050	266	28	1	0	0
9	0	1	255	3025	7770	6951	2646	462	36	1	0
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

It is clear from the recursive definition that the $S(n,k)$ are integers, that $S(n,n)=1$ and that $S(n,k)=0$ if $k>n$. Interesting problem: prove directly that $\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n=0$ whenever $k>n>0$.



Plot of $\log S(n,k)+1$ for $k=5,10,\ldots$ and $n=1,2,\ldots,300$

Euler–Maclaurin summation

Let $f(x)$ be a sufficiently well-behaved function. Suppose we want to sum $f(x)$ over integer values of x and that we know how to integrate $f(x)$. Let $m < n$ be integers. Then

$$\sum_{k=m+1}^{n} f(r) \;=\; \int_m^n f(x)\,dx + \sum_{k=1}^{r} \left(f^{(k-1)}(n) - f^{(k-1)}(m)\right) \frac{\widehat{B}_k}{k!} - \int_m^n f^{(r)}(x) \frac{\widehat{B}_r(\lfloor x\rfloor - x)}{r!}\,dx,$$

or, making use of the fact that $B_n = 0$ for odd $n \geq 3$,

$$\begin{aligned}\sum_{k=m+1}^{n} f(k) \;&=\; \int_m^n f(x)\,dx + \frac{f(n)-f(m)}{2} + \sum_{k=1}^{s} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) \frac{B_{2k}}{(2k)!} \\ &\quad - \int_m^n f^{(2r)}(x) \frac{B_{2r}(x - \lfloor x\rfloor)}{(2r)!}\,dx, \qquad (r \geq 1).\end{aligned}$$

If $f(x)$ is a polynomial, the last term will vanish for sufficiently large r. For example, to get the formula for the sum of the first n squares put $m = 0$ and $f(x) = x^2$:

$$\sum_{k=1}^{n} k^2 \;=\; \int_0^n x^2\,dx + \frac{n^2}{2} + 2n\,\frac{B_2}{2!} \;=\; \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \;=\; \frac{n(2n+1)(n+1)}{6}.$$

Poly-Bernoulli numbers

If we define the more general *poly-Bernoulli numbers* $B_n^{(k)}$ by

$$\mathrm{Li}_k(t) \;=\; \sum_{m=1}^{\infty} \frac{t^m}{m^k}, \qquad \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \;=\; \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

then $k = 1$ gives $\mathrm{Li}_1(x) = -\log(1-x)$ and the generating function $t/(1-e^{-t})$ of $B_n^{(1)} = \widehat{B}_n$.

Properties of the Bernoulli numbers

Theorem 1 (Explicit formula for the Bernoulli numbers) *We have*

$$B_n \;=\; \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n \;=\; \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\, k!\, S(n,k). \tag{3}$$

Proof We refer to the generating function:

$$\frac{x}{e^x-1} \;=\; \frac{\log(1+(e^x-1))}{e^x-1} \;=\; \frac{1}{e^x-1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(e^x-1)^k}{k} \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{(e^x-1)^k}{k+1}.$$

Now expand the binomial $(e^x-1)^k$ and then expand e^x:

$$\begin{aligned}\frac{x}{e^x-1} \;&=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{jx} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=0}^{\infty} \frac{j^n x^n}{n!} \\ &=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \\ &=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} k! S(n,k).\end{aligned}$$

But $S(n,k)=0$ when $k>n$. Hence

$$\frac{x}{e^x-1} \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n}\frac{(-1)^k}{k+1}k!S(n,k),$$

as required to get the right-hand expression in (3). The other expression follows by substituting for $S(n,k)$. ☐

Theorem 2 (von Staudt – Clausen) *If n is a positive even integer, then*

$$B_n + \sum_{p-1|n,\ p\ prime}\frac{1}{p} \;\equiv\; 0 \pmod 1. \tag{4}$$

Proof Suppose $n \geq 2$ is even. We consider each of the terms in the sum over k in either of the two expressions of (3).

If $k+1$ is not prime and $k>3$, then $k!/(k+1)$ is an integer, as also is $S(n,k)$; so the term corresponding to this k is an integer. Also one can verify directly that the terms corresponding to $k=0$ and $k=3$ are integers.

Now suppose $k+1$ is prime and k does not divide n. Write $n=qk+r$ with $0<r<k$. Then for $0<j\leq k$, $j^{qk}\equiv 1 \pmod{k+1}$. Hence modulo $k+1$ the sum over j becomes $\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^r = \pm k!S(r,k)=0$ since $k>r$.

Finally, suppose $k+1$ is prime and k divides n. Then for $0<j\leq k$, $j^n\equiv 1 \pmod{k+1}$, and $0^n=0$. Therefore $\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n \equiv -1 \pmod{k+1}$ and hence modulo 1 there is a contribution of $-1/(k+1)$ to the sum. ☐

When n is odd the sum is zero modulo 1; the $k=1$ term contributes $1/2$, as before, but now the $k=3$ term also contributes $1/2$. All other k terms are integers.

The Von Staudt–Clausen theorem allows you to compute the fractional part of B_n for some quite large n. For example, if $n=2^{8000000000}$, the divisors of n are powers of two and the denominators of the fractions in (4) are just 2 and the Fermat primes, 3, 5, 17, 257 and 65537. On the other hand, the calculation is currently impossible for, say, $n=2^{9000000000}$.

It is convenient to have an expression for the absolute value of B_n. So we rearrange (4) to get, writing $\{x\}$ for $x-\lfloor x\rfloor$,

$$|B_n| \;=\; \lfloor |B_n| \rfloor + \begin{cases} \left\{\sum_{p-1|n,\ p\ \text{prime}} 1/p\right\}, & n>0,\ n\equiv 0 \pmod 4,\\ 1-\left\{\sum_{p-1|n,\ p\ \text{prime}} 1/p\right\}, & n\equiv 2 \pmod 4,\\ 0, & n=0,\\ 1/2, & n=1,\\ 0, & n>1,\ n \text{ odd}. \end{cases} \tag{5}$$

Thus for $n=2,4,\ldots,$ we get

$$\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\frac{691}{2730},\frac{1}{6},\frac{47}{510},\frac{775}{798},\frac{41}{330},\frac{17}{138},\frac{691}{2730},\frac{1}{6},\frac{59}{870},\frac{12899}{14322},$$

$$\frac{47}{510},\frac{1}{6},\frac{638653}{1919190},\frac{1}{6},\frac{2011}{13530},\frac{1}{1806},\frac{53}{690},\frac{41}{282},\frac{14477}{46410},\frac{5}{66},\frac{83}{1590},\frac{775}{798},\frac{59}{870},\frac{53}{354},\frac{22298681}{56786730},$$

$$\frac{1}{6},\frac{47}{510},\frac{62483}{64722},\frac{1}{30},\frac{289}{4686},\frac{48540859}{140100870},\frac{1}{6},\frac{1}{30},\frac{37}{3318},\frac{47717}{230010},$$

$$\frac{77}{498},\frac{1058237}{3404310},\frac{1}{6},\frac{5407}{61410},\frac{230759}{272118},\frac{77}{1410},\frac{1}{6},\frac{1450679}{4501770},\frac{1}{6},\frac{4471}{33330}$$

Theorem 3 *The Bernoulli numbers are related to the Riemann zeta-function by*

$$\widehat{B}_n \;=\; -\,n\zeta(1-n), \quad n \ge 0; \qquad B_n \;=\; -\,\frac{2\,n!}{(2\pi i)^n}\,\zeta(n), \quad \textit{even } n \ge 0. \tag{6}$$

Proof (Titchmarsh [6, section 2.4]) Let s be a complex veriable with $\operatorname{Re} s > 1$. Using

$$\int_0^\infty x^{s-1} e^{-kx}\,dx \;=\; \frac{\Gamma(s)}{k^s}$$

we get

$$\Gamma(s)\zeta(s) \;=\; \sum_{k=1}^\infty \int_0^\infty x^{s-1} e^{-ks}\,dx \;=\; \int_0^\infty x^{s-1} \sum_{k=1}^\infty e^{-kx}\,dx \;=\; \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

Let C_ρ, $\rho > 0$, denote the contour that goes from $+\infty$ to ρ on the real axis, circles the origin once anticlockwise and then returns to $+\infty$. Let $\rho \to 0$. Since the integral along the circle tends to zero as ρ tends to zero, we have

$$\int_{C_0} \frac{z^{s-1}}{e^z - 1}\,dz \;=\; \int_\infty^0 \frac{z^{s-1}}{e^z-1}\,dz + \int_0^\infty \frac{(e^{2\pi i} z)^{s-1}}{e^z - 1}\,dz \;=\; (e^{2\pi i s} - 1)\Gamma(s)\zeta(s).$$

Using the well-known formula $\Gamma(1-z)\Gamma(z) = \pi/(\sin \pi z)$, we obtain

$$\zeta(s) \;=\; \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} \int_{C_0} \frac{z^{s-1}}{e^z - 1}\,dz \;=\; \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} \int_{C_0} \frac{z^{s-1}}{e^z - 1}\,dz,$$

which extends the domain of $\zeta(s)$ to the whole complex plane except for $s = 1$. Now let $n \ge 1$ be an integer and put $s = 1 - n$. Then

$$\zeta(1-n) \;=\; \frac{(-1)^{n-1}(n-1)!}{2\pi i} \int_{C_0} \frac{z^{-n}}{e^z - 1}\,dz \;=\; \frac{(-1)^{n-1}(n-1)!}{2\pi i} \int_{C_0} \sum_{k=0}^\infty \frac{B_k}{k!}\, z^{k-n-1}\,dz.$$

But the value of integral on the right is $2\pi i$ times the residue of the pole at $z = 0$, and this is just the coefficient of $1/z$ in the sum. Hence, recalling that $B_1 = -\widehat{B}_1$ and $B_n = 0$ for odd $n \ge 3$,

$$\zeta(1-n) \;=\; \frac{(-1)^{n-1}(n-1)!}{2\pi i} \cdot 2\pi i\, \frac{B_n}{n!} \;=\; \frac{(-1)^{n-1} B_n}{n} \;=\; -\,\frac{\widehat{B}_n}{n}.$$

This is the left-hand equality in (6) for $n \ge 1$. For the remaining case, recall that

$$\zeta(s) \;=\; \frac{1}{s-1} - \gamma + O(s-1) \qquad \text{as } s \to 1.$$

The right-hand equality follows from the functional equation of $\zeta(s)$,

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) \;=\; \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s).$$

(See Edwards [1], for example.) □

Of course we can discard the sign if we are interested only in the absolute value:

$$|B_n| \;=\; \frac{2\,n!}{(2\pi)^n}\,\zeta(n) \;=\; \frac{2\,n!}{(2\pi)^n} \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\right), \quad \text{even } n \ge 2. \tag{7}$$

We can use (5) and (7) to determine $|B_n|$ exactly. Write $b(n) = (2\,n!)^{1/n}/(2\pi)$. Then split the sum into three parts,

$$\begin{aligned} |B_n| &= I_n + E_n + R_n, \\ I_n &= \left\lfloor \sum_{k=1}^{\lceil b(n)\rceil} \left(\frac{b(n)}{k}\right)^n \right\rfloor, \quad E_n = \sum_{k=1}^{\lceil b(n)\rceil} \left(\frac{b(n)}{k}\right)^n - I_n, \quad R_n = \sum_{k=\lceil b(n)+1\rceil}^{\infty} \left(\frac{b(n)}{k}\right)^n. \end{aligned}$$

We compute I_n exactly and E_n approximately, we assume that R_n is small enough to be ignored, and we get the exact value of F_n, the fractional part of $|B_n|$, using the von Staudt – Clausen formula, (5). Then for positive even n, we have $|B_n| = I_n + \epsilon + F_n$, where $\epsilon = 0$ if $E_n < F_n$, $\epsilon = 1$ otherwise. Example:

$$\lceil b(1000)\rceil = 59, \quad I_{1000} = 5318704469\ldots6955251716 \approx 5.318704469415522 \times 10^{1769},$$

$$E_{1000} \approx 0.1625177182 \quad \text{and} \quad F_{1000} = \frac{55743421}{342999030} \approx 0.1625177220 > E_{1000}.$$

There is an interesting application concerning prime generating functions, [4], and also an interesting problem. Is there an n that requires $\epsilon = 1$? By simple-minded direct calculation there are no instances for even n up to 3200. But could it possibly happen that E_n is very nearly equal to but less than 1, F_n is very small, and R_n, which is very small anyway, is just large enough so that $E_n + R_n > 1$? Then we would have $F_n = E_n + R_n - 1$ and hence $\lfloor |B_n| \rfloor = I_n + 1$. A similar problem is presented in *M500* [2, 5].

References

[1] H. M. Edwards, *Riemann's Zeta-Function.*

[2] TF, Problem 267.4 – Bernoulli numbers, *M500* **267** (December 2015).

[3] A. A. Lovelace, Note G in: L. F. Menabrea, Sketch of The Analytical Engine Invented by Charles Babbage With notes upon the Memoir by the Translator, Ada Augusta, Countess of Lovelace, *Bibliothèque Universelle de Genève* **82** (October 1842).

[4] R. Thompson, Bernoulli numbers and prime generating functions, *M500* **267** (December 2015).

[5] R. Thompson, Solution 267.4 – Bernoulli numbers, *M500*, to appear.

[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function.*