Poncelet's Porism
Talk given at LSBU, April 2015

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Statement of the theorem

Theorem 1 (Jean–Victor Poncelet, 1788–1867) *Let $\mathcal{C}$ and $\mathcal{D}$ be two plane conics. Given $n \geq 3$, if it is possible to find an n-gon which is circumscribed by $\mathcal{C}$ and inscribed by $\mathcal{D}$, then every point of $\mathcal{C}$ is a vertex of one such polygon.*

Proof See [1].

Theorem 2 (Two circles; see [2]) *Let $\mathcal{C}$ be a circle of radius R with centre $(0,0)$, and let $\mathcal{D}$ be a circle of radius r with centre $(d,0)$, $R > r > 0$, $R > d > 0$. Let $n \geq 3$ be an integer. Let*

$$a = \frac{1}{R+d}, \qquad b = \frac{1}{R-d}, \qquad c = \frac{1}{r},$$

$$\lambda = 1 + \frac{2c^2(a^2-b^2)}{a^2(b^2-c^2)}, \qquad \omega = \cosh^{-1}\lambda, \qquad k^2 = 1 - \exp(-2\omega).$$

If a convex (presumably—otherwise I can't get this to work) n-gon exists with circum-circle $\mathcal{C}$ and in-circle $\mathcal{D}$, then

$$\mathrm{sc}\left(\frac{K(k)}{n},\ k\right) = \frac{c\sqrt{b^2-a^2} + b\sqrt{c^2-a^2}}{a(b+c)}. \tag{1}$$

Proof omitted.

Here, $K(k)$ is the complete elliptic function of the first kind,

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

and $\mathrm{sc}(z,k)$ is the Jacobian elliptic function defined by

$$\mathrm{sc}(z,k) = \tan\varphi, \quad \text{where} \quad z = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

When $d = 0$, the condition (1) reduces to $r = R\cos(\pi/n)$ since $k = 0$, $\mathrm{sc}(z,0) = \tan z$, and (1) becomes

$$\mathrm{sc}\left(\frac{K(0)}{n}, 0\right) = \tan\frac{\pi}{2n} = \frac{\sqrt{R^2-r^2}}{R+r}.$$

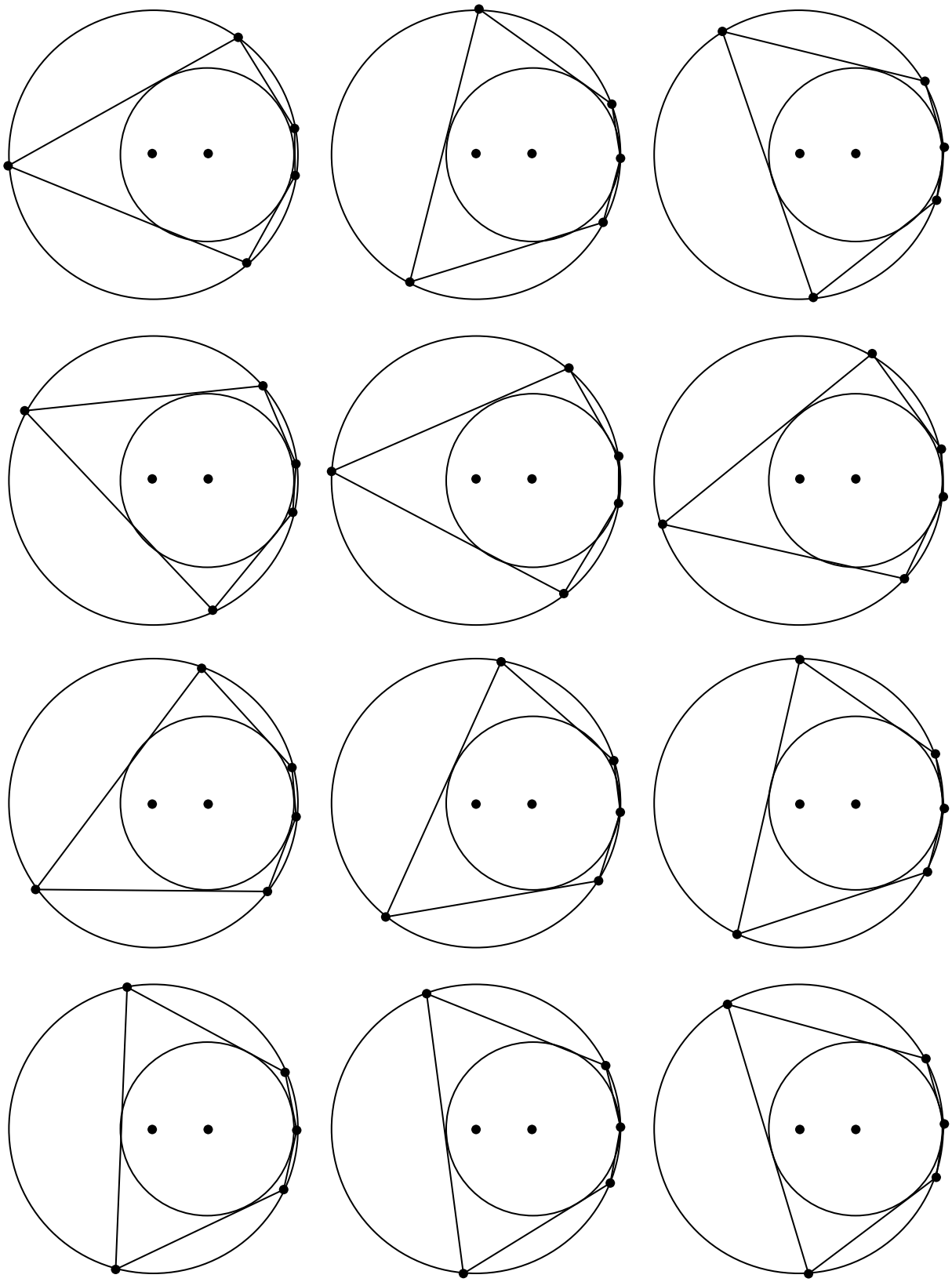
Here is what happens when $R = 10$, $r = 6$ and $n = 5$. Noting that

$$\mathrm{sc}\left(\frac{K(k)}{n}, k\right) = w \quad \Rightarrow \quad \frac{K(k)}{n} = \int_0^{\arctan w} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

we can solve (1) for d and find that $d \approx 3.86702$ is a valid displacement for $\mathcal{D}$ with respect to $\mathcal{C}$. Also

$$\lambda \approx 192.743, \qquad \omega \approx 5.9545, \qquad k^2 = 0.999993, \qquad \mathrm{sc}\left(\frac{K(k)}{5},\ k\right) \approx 2.05546.$$

Now we can draw convex pentagons starting at any point on $\mathcal{C}$.



Cayley's solution [3]

Theorem 3 *Let $\mathcal{C}[x,y,z]=0$ and $\mathcal{D}[x,y,z]=0$ be the homogeneous quadratic equations in projective coordinates that define suitable conics $\mathcal{C}$ and $\mathcal{D}$. Let Q be the 3×3 symmetric matrix of the quadratic form $t\mathcal{C}[x,y,z]+\mathcal{D}[x,y,z]$ and let A_i be defined by*

$$\sqrt{\det(t\mathcal{C}+\mathcal{D})} \;=\; \sqrt{\det Q} \;=\; A_0 + A_1 t + A_2 t^2 + \dots . \tag{2}$$

Let $n \geq 3$. Then the condition for the existence of an n-gon circumscribed by $\mathcal{C}$ and inscribed by $\mathcal{D}$ is

$$\begin{aligned} \det\begin{bmatrix} A_2 & \dots & A_{m+1} \\ \dots & \dots & \dots \\ A_{m+1} & \dots & A_{2m} \end{bmatrix} &= 0 \quad \textit{for odd } n=2m+1, \\ \det\begin{bmatrix} A_3 & \dots & A_{m+1} \\ \dots & \dots & \dots \\ A_{m+1} & \dots & A_{2m-1} \end{bmatrix} &= 0 \quad \textit{for even } n=2m. \end{aligned} \tag{3}$$

Proof See [3]. See (ii) below for some idea of what it means to be suitable.

Notes

(i) The matrix for $ax^2+by^2+cz^2+2dxy+2exz+2fyz$ is $\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$.

(ii) For the proof in [3] to work, (2) must represent an elliptic curve. Moreover, the curve must avoid the origin; otherwise there will be trouble with the expansion on the right of (2). So the determinant of the matrix of $t\mathcal{C}+\mathcal{D}$ must be a cubic in t with distinct non-zero roots. The distinctness of the roots may be relaxed as is the case, for example, where the conics are concentric circles, for then one can argue by continuity.

(iii) The inscribing of the n-gon by $\mathcal{D}$ may be interpreted quite liberally. We allow situations where a side of the polygon needs to be extended to get to the tangent point where it meets $\mathcal{D}$.

(iv) One may interpret an n-gon traversed k times as a kn-gon. So if the condition holds for n, it must hold for $2n$, $3n$, $\dots$. Take the simple case where $n=3$ and $\mathcal{C}$ and $\mathcal{D}$ are circles with centre $[0,0,1]$ and radii 2 and 1 respectively. Then $\mathcal{C}=x^2+y^2-4z^2$, $\mathcal{D}=x^2+y^2-z^2$ and

$$\begin{aligned} \sqrt{\det(t\mathcal{C}+\mathcal{D})} &= \sqrt{\det\begin{bmatrix} t+1 & 0 & 0 \\ 0 & t+1 & 0 \\ 0 & 0 & -4t-1 \end{bmatrix}} \;=\; i(t+1)\sqrt{4t+1} \\ &= i(1+3t+2t^3-6t^4+18t^5-56t^6+180t^7-594t^8+\dots). \end{aligned}$$

We can see straight away that the condition for $n=3$ reduces to $\det[A_2]=0$. Moreover, for $n=6$ we have

$$\det\begin{bmatrix} A_3 & A_4 \\ A_4 & A_5 \end{bmatrix} \;=\; -\det\begin{bmatrix} 2 & -6 \\ -6 & 18 \end{bmatrix} \;=\; 0$$

and when $n=9$

$$\det\begin{bmatrix} A_2 & A_3 & A_4 & A_5 \\ A_3 & A_4 & A_5 & A_6 \\ A_4 & A_5 & A_6 & A_7 \\ A_5 & A_6 & A_7 & A_8 \end{bmatrix} \;=\; \det\begin{bmatrix} 0 & 2 & -6 & 18 \\ 2 & -6 & 18 & -56 \\ -6 & 18 & -56 & 180 \\ 18 & -56 & 180 & -594 \end{bmatrix} \;=\; 0,$$

and so on. Is there a simple proof that these determinants are always zero?

Example 1: Two ellipses and a triangle Let

$$\mathcal{C}[x,y,z] \;=\; x^2+3y^2-2xy-\frac{1}{2}yz-z^2, \quad \mathcal{D}[x,y,z] \;=\; 3x^2+y^2-\frac{3}{16}z^2.$$

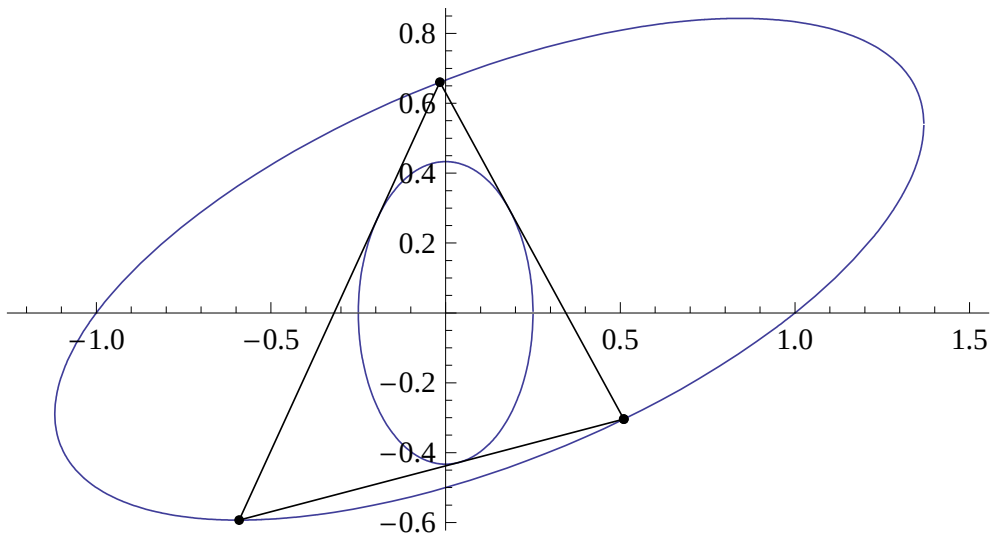
Then

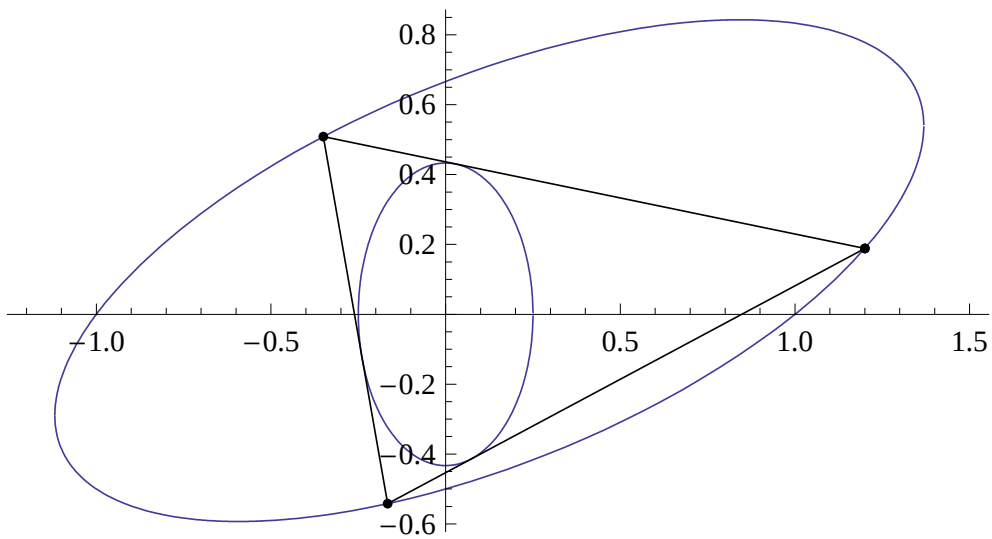
$$\det Q \;=\; \det\begin{bmatrix} t+3 & -t & 0 \\ -t & 3t+1 & -t/4 \\ 0 & -t/4 & -t-3/16 \end{bmatrix} \;=\; \frac{1}{16}(-9-78t-169t^2-33t^3)$$

and when we expand

$$\sqrt{\det Q} \;=\; \frac{3i}{4}+\frac{13it}{4}+\frac{11it^3}{8}-\frac{143it^4}{24}+\frac{1859it^5}{72}-\frac{97757it^6}{864}+\frac{1299155it^7}{2592}-\frac{8720569it^8}{3888}+\ldots$$

we see that the t^2 term is missing; so $\det[A_2]=0$ and the condition for a triangle circumscribed by $\mathcal{C}$ and inscribed by $\mathcal{D}$ is satisfied. A quadrilateral does not exist because $A_3 \neq 0$. However, a degenerate $3k$-gon exists; so the determinants for $n=3k$ must all be zero. For example, you can verify that $\det\begin{bmatrix} A_3 & A_4 \\ A_4 & A_5 \end{bmatrix} = \det\begin{bmatrix} A_2 & A_3 & A_4 & A_5 \\ A_3 & A_4 & A_5 & A_6 \\ A_4 & A_5 & A_6 & A_7 \\ A_5 & A_6 & A_7 & A_8 \end{bmatrix} = 0.$





Example 2: Two ellipses and an octagon Let

$$\mathcal{C}[x,y,z] \;=\; x^2+4y^2-\frac{12}{5}\,xy-\frac{1}{2}\,yz-z^2, \quad \mathcal{D}[x,y,z] \;=\; 3x^2+5y^2-r^2z^2.$$

Then it turns out that if $r \approx\ 0.552345421970$, we can construct an octagon with circumconic $\mathcal{C}$ and in-conic $\mathcal{D}$. So

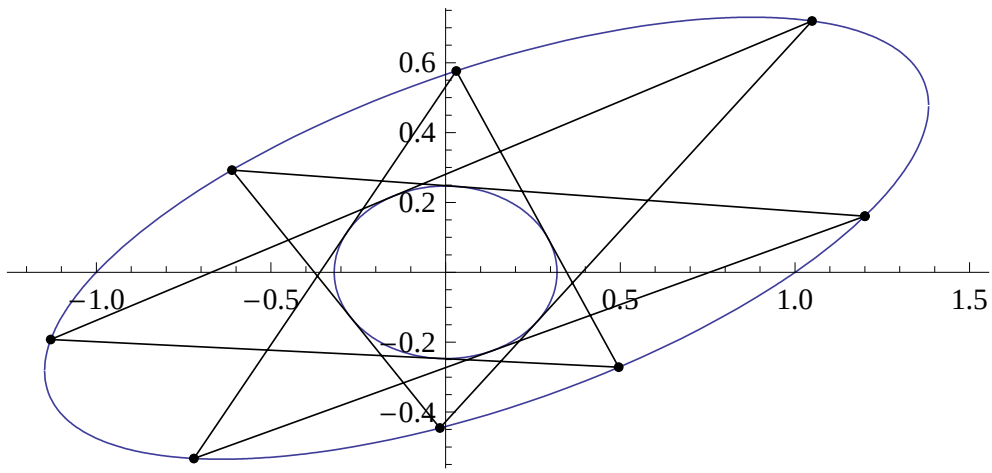
$$\det Q \;=\; \det\begin{bmatrix} t+3 & -6t/5 & 0 \\ -6t/5 & 4t+5 & -t/4 \\ 0 & -t/4 & -t-r^2 \end{bmatrix} \;=\; -15r^2-15t-17r^2t-\frac{275t^2}{16}-\frac{64r^2t^2}{25}-\frac{1049t^3}{400}.$$

To avoid horrendous algebraic manipulations we substitute for r and work approximately:

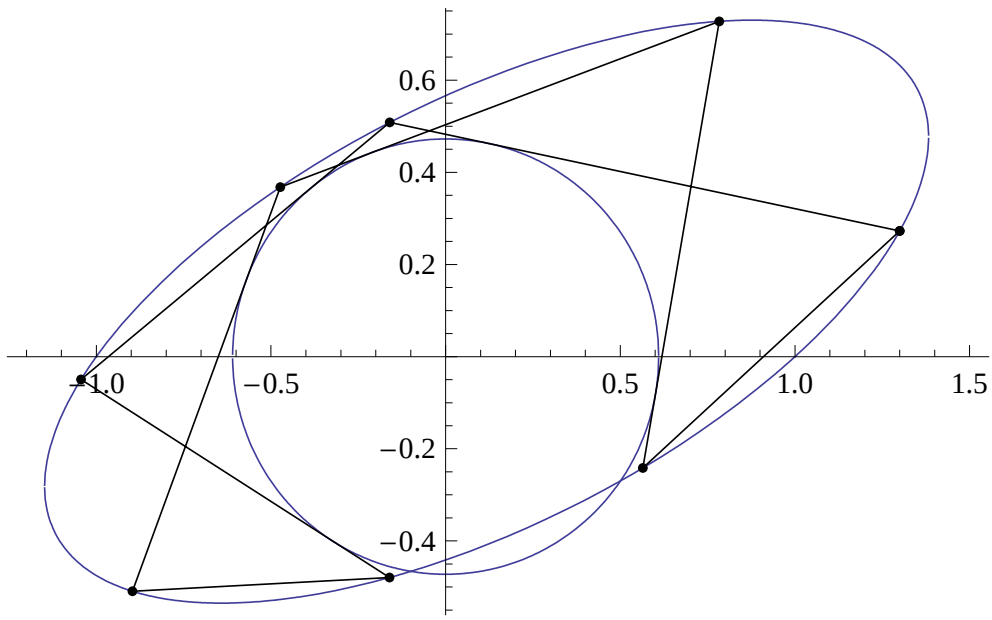
$$\begin{aligned}\sqrt{\det Q} \;\approx\; & 2.13922i+4.71817it-1.00331it^2+2.82581it^3-6.46775it^4+15.5903it^5 \\ & -39.285it^6+102.501it^7+O(t^8),\end{aligned}$$

and for $n=8$ we check the A matrix condition. Indeed:

$$\det\begin{bmatrix} A_3 & A_4 & A_5 \\ A_4 & A_5 & A_6 \\ A_5 & A_6 & A_7 \end{bmatrix} \;\approx\; \det\begin{bmatrix} 2.82581i & -6.46775i & 15.5903i \\ -6.46775i & 15.5903i & -39.285i \\ 15.5903i & -39.285i & 102.501i \end{bmatrix} \;\approx\; 0.$$



The A determinant is zero also for $r \approx 1.0561374780$.



Example 3: Concentric circles Let

$$\mathcal{C}[x,y,z] \;=\; x^2+y^2-z^2, \quad \mathcal{D}[x,y,z] \;=\; x^2+y^2-r^2z^2.$$

with $r>0$. Then (note the repeated factor $t+1$)

$$\det Q \;=\; \det\begin{bmatrix} t+1 & 0 & 0 \\ 0 & t+1 & 0 \\ 0 & 0 & -t-r^2 \end{bmatrix} \;=\; -(t+1)^2(t+r^2);$$

$$\sqrt{\det Q} \;=\; i(t+1)r\sum_{k=0}^{\infty}\binom{1/2}{k}\frac{t^k}{r^{2k}};$$

$$A_0 \;=\; ir, \quad A_1 = i\,\frac{2r^2+1}{2r}, \quad A_2 = i\,\frac{4r^2-1}{8r^3},$$

$$A_k \;=\; \frac{ir}{r^{2k}}\binom{1/2}{k}+\frac{ir}{r^{2k-2}}\binom{1/2}{k-1} \;=\; \frac{i(-1)^k}{2^{2k-3}r^{2k-1}}\,\frac{(2kr^2-2k+3)(2k-5)!}{(k-3)!k!}, \quad k=3,4,\ldots.$$

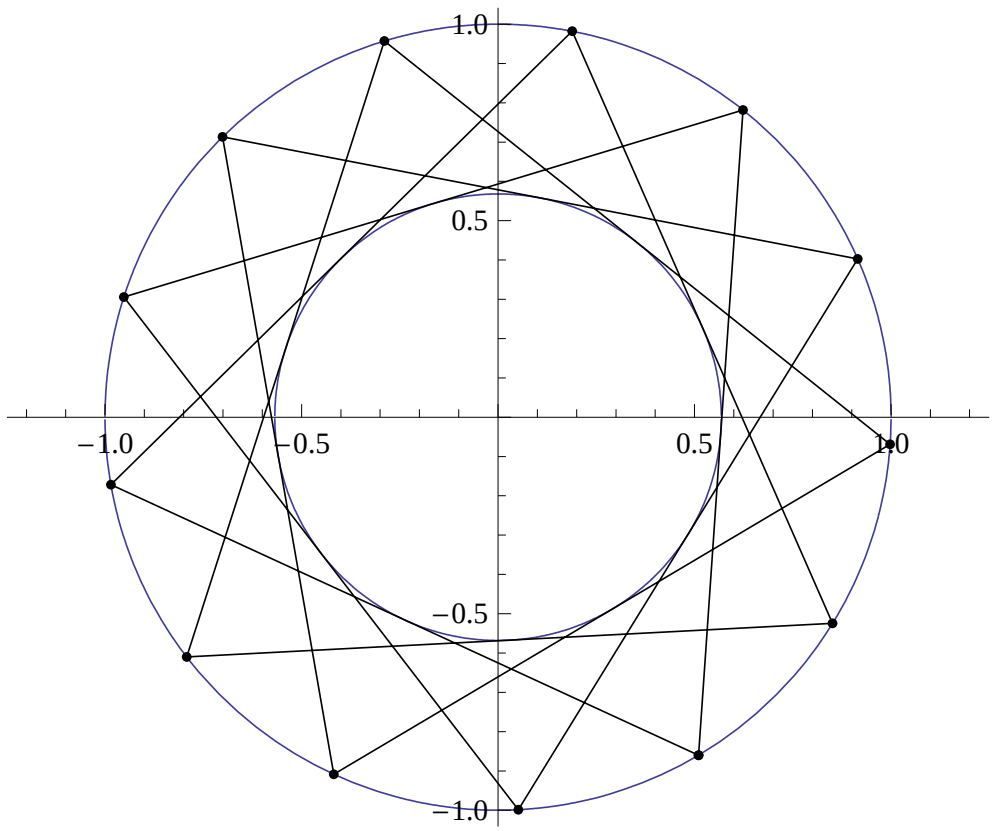
Immediately we see that $r=1/2=\cos(\pi/3)$ implies $\det[A_2]=0$ and a triangle exists with circum-circle $\mathcal{C}$ and in-circle $\mathcal{D}$. Also $\det[A_3]=(2r^2-1)\times$ (something) and so, as expected, a square exists when $r=\sqrt{2}/2=\cos(\pi/4)$. To deal $n=5$ we calculate

$$\det\begin{bmatrix} A_2 & A_3 \\ A_3 & A_4 \end{bmatrix} \;=\; (-1+2r+4r^2)(-1-2r+4r^2)(\text{something}),$$

and the biquadratic has two positive roots, $r=(\sqrt{5}\pm 1)/4$, $\cos(\pi/5)$ and $\cos(2\pi/5)$ for the two types of pentagon. For $n=6$, we have

$$\det\begin{bmatrix} A_3 & A_4 \\ A_4 & A_5 \end{bmatrix} \;=\; (-1+2r)(1+2r)(-3+4r^2)(\text{something}).$$

The positive roots of the quartic are $r=1/2$ the triangle again, and $r=\sqrt{3}/2=\cos(\pi/6)$, a hexagon. When we calculate the determinant of the appropriate A matrix for larger values of n we should get a polynomial amongst whose roots are $|\cos(2\pi k/n)|$, $\gcd(k,n)=1$, $k=1,\,2,\,\ldots,\,\lfloor n/2\rfloor$. However, this does not seem (to me) to be immediately obvious.



Example 4: Hyperbola and ellipse Let

$$\mathcal{C}[x,y,z] \;=\; x^2-4y^2-2xy-\frac{yz}{2}-z^2, \quad \mathcal{D}[x,y,z] \;=\; 3x^2+5y^2-r^2z^2.$$

with $r>0$. Then

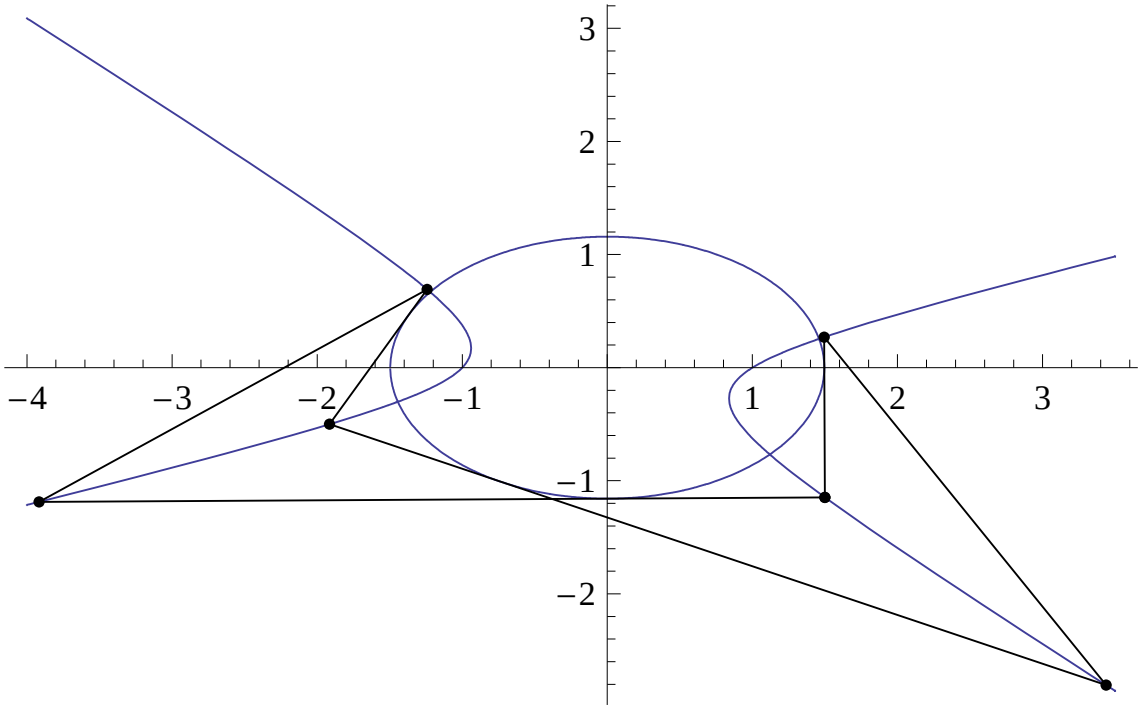
$$\det Q \;=\; \det\begin{bmatrix} t+3 & -t & 0 \\ -t & 5-4t & -t/4 \\ 0 & -t/4 & -t-r^2 \end{bmatrix} \;=\; -15r^2-15t+7r^2t+\frac{109t^2}{16}+5r^2t^2+\frac{79t^3}{16}.$$

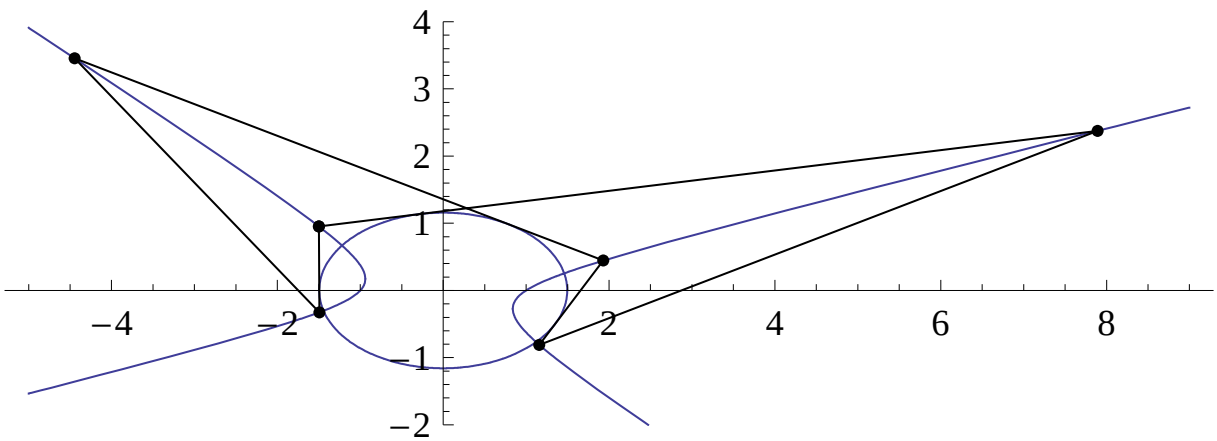
Putting $r \approx 2.58997822431$ gives

$$\sqrt{\det Q} \;\approx\; 10.0309i-1.59287it-2.13787it^2-0.585597it^3-0.320809it^4-0.17575it^5+O(t^6)$$

and for $n=6$ we check the A matrix condition. Indeed:

$$\det\begin{bmatrix} A_3 & A_4 \\ A_4 & A_5 \end{bmatrix} \;\approx\; \det\begin{bmatrix} -0.585597i & -0.320809i \\ -0.320809i & -0.175750i \end{bmatrix} \;\approx\; 0.$$





Example 5: Hyperbola and parabola: Let

$$\mathcal{C}[x,y,z] \;=\; x^2 - 4y^2 - 2xy - \frac{yz}{2} - z^2, \quad \mathcal{D}[x,y,z] \;=\; x^2 + yz - r^2 z^2.$$

with $r > 0$. Then

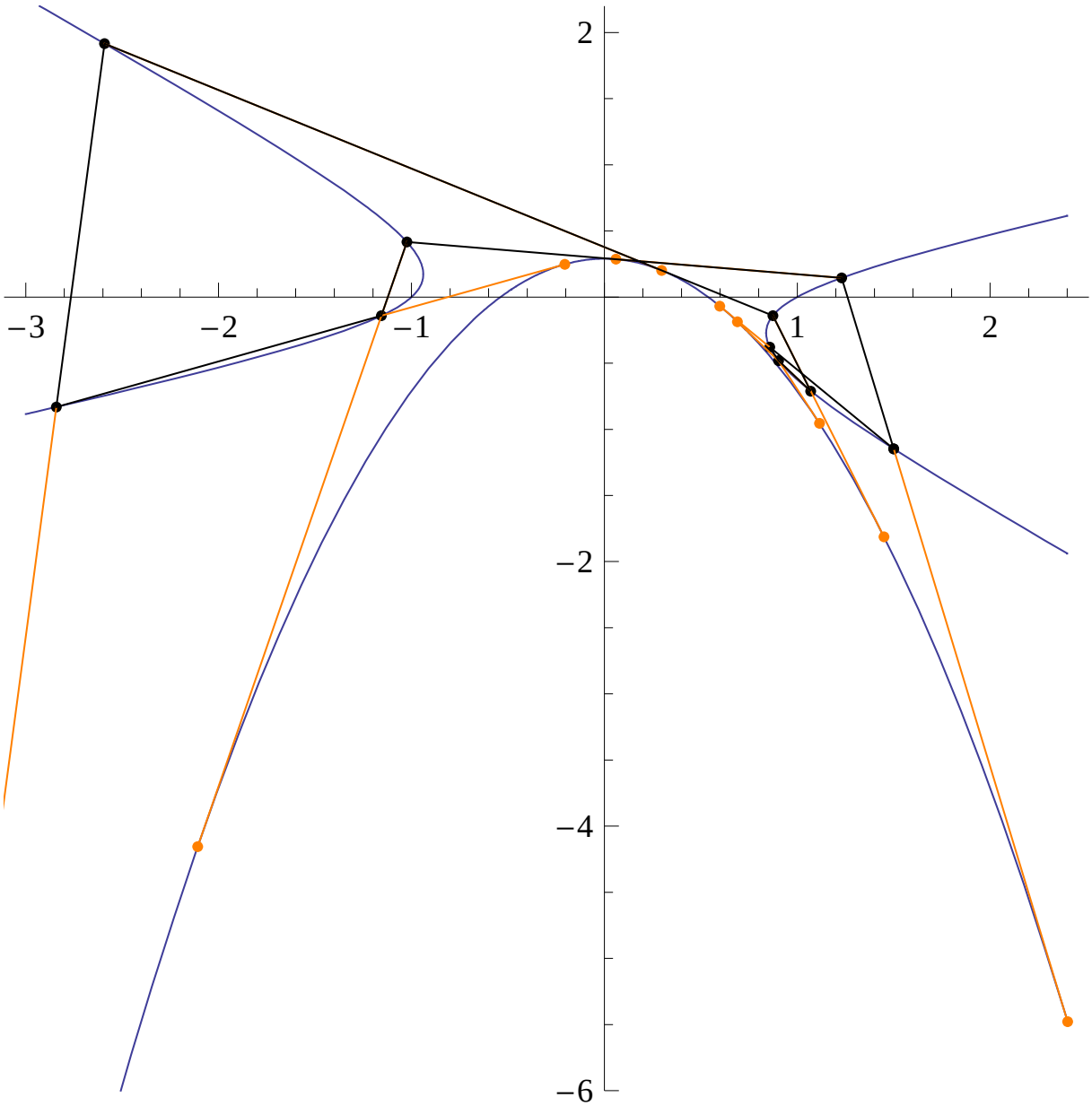
$$\det Q \;=\; \det \begin{bmatrix} t+1 & -t & 0 \\ -t & 4t & 1/2 - t/4 \\ 0 & 1/2 - t/4 & -t - r^2 \end{bmatrix} \;=\; -\frac{1}{4} + 4r^2 t + \frac{67t^2}{16} + 5r^2 t^2 + \frac{79t^3}{16}.$$

Putting $r \approx 0.5382016380$ gives

$$\begin{aligned}\sqrt{\det Q} \;\approx\; & 0.5i - 1.15864it - 6.97826it^2 - 21.1081it^3 - 97.6098it^4 - 520.786it^5 \\ & - 3014.66it^6 - 18374.9it^7 - 116167it^8 - 754579it^9 + O(t^{10})\end{aligned}$$

and for rather contorted decagon we check the A matrix condition for $n = 10$. Indeed:

$$\det \begin{bmatrix} A_3 & A_4 & A_5 & A_6 \\ A_4 & A_5 & A_6 & A_7 \\ A_5 & A_6 & A_7 & A_8 \\ A_6 & A_7 & A_8 & A_9 \end{bmatrix} \;\approx\; \det \begin{bmatrix} -21.1081i & -97.6098i & -520.786i & -3014.66i \\ -97.6098i & -520.786i & -3014.66i & -18374.9i \\ -520.786i & -3014.66i & -18374.9i & -116167i \\ -3014.66i & -18374.9i & -116167i & -754579i \end{bmatrix} \;\approx\; 0.$$



Example 6: Two hyperbolas: Let

$$\mathcal{C}[x,y,z] \;=\; x^2 - 3y^2 - xy - \frac{1}{2}yz - z^2, \quad \mathcal{D}[x,y,z] \;=\; -3x^2 + 2y^2 + \frac{1}{5}yz - r^2z^2.$$

with $r > 0$. Then

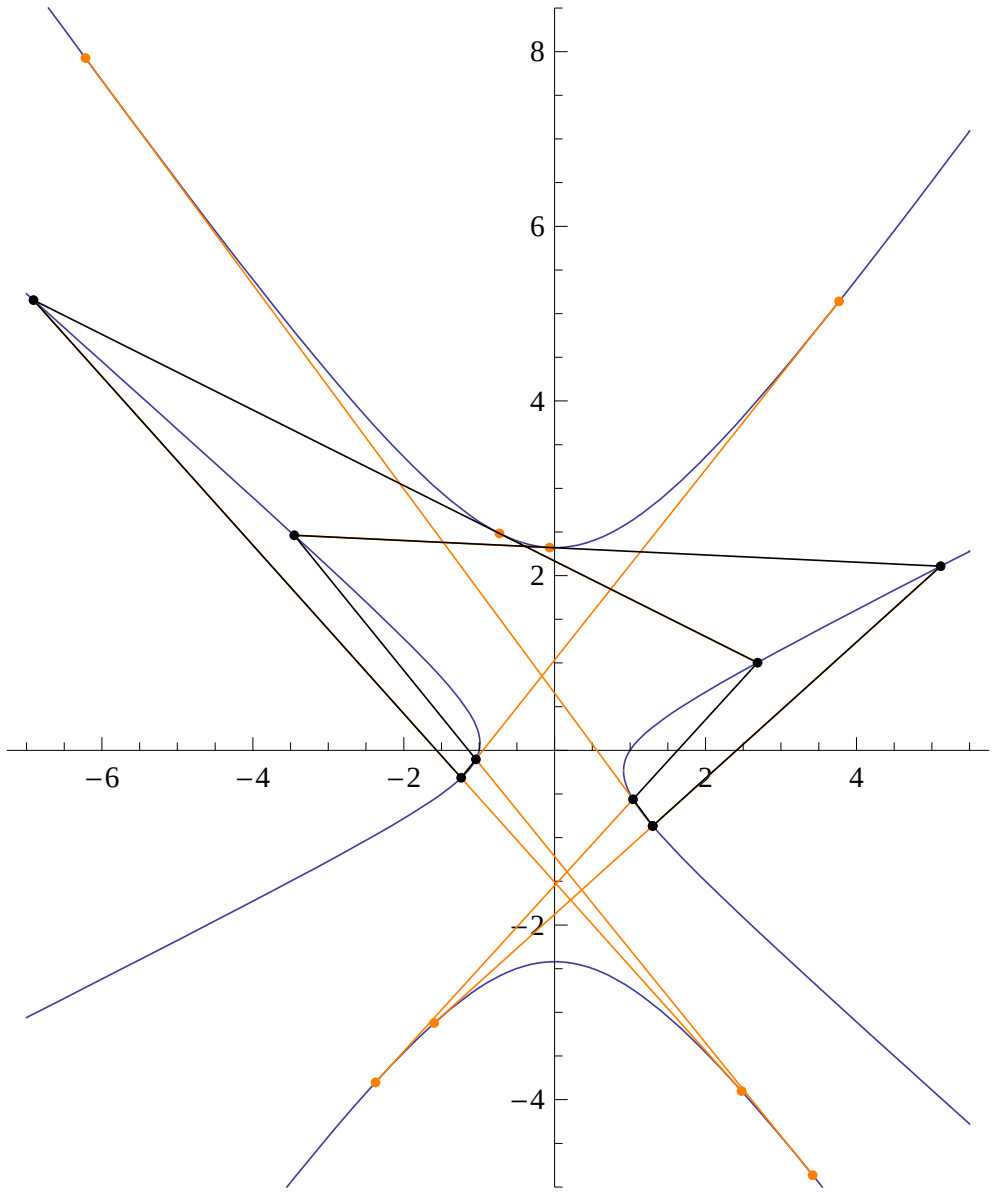
$$\begin{aligned}\det Q \;&=\; \det\begin{bmatrix} t-3 & -t/2 & 0 \\ -t/2 & 2-3t & 1/10 - t/4 \\ 0 & 1/10 - t/4 & -t - r^2 \end{bmatrix} \\ &=\; \frac{3}{100} + 6r^2 + \frac{146t}{25} - 11r^2t - \frac{861t^2}{80} + \frac{13r^2t^2}{4} + \frac{51t^3}{16}.\end{aligned}$$

Putting $r \approx 3.35162989586$ gives

$$\begin{aligned}\sqrt{\det Q} \;\approx\; & 8.21161 - 7.16837t - 1.56116t^2 - 1.16874t^3 - 1.16866t^4 - 1.24239t^5 \\ & - 1.3899t^6 - 1.61585t^7 + O(t^8)\end{aligned}$$

and for an octagon we check the A matrix condition for $n = 8$. Indeed:

$$\det\begin{bmatrix} A_3 & A_4 & A_5 \\ A_4 & A_5 & A_6 \\ A_5 & A_6 & A_7 \end{bmatrix} \;\approx\; \det\begin{bmatrix} -1.16874 & -1.16866 & -1.24239 \\ -1.16866 & -1.24239 & -1.3899 \\ -1.24239 & -1.3899 & -1.61585 \end{bmatrix} \;\approx\; 0.$$



Example 7: Two circles, revisited: Let

$$\mathcal{C}[x,y,z] \;=\; x^2+y^2-R^2z^2, \quad \mathcal{D}[x,y,z] \;=\; (x-d)^2+y^2-r^2z^2.$$

with $R>r>0$ and $d>0$. Then

$$\det Q \;=\; \det\begin{bmatrix} t+1 & 0 & -d \\ 0 & t+1 & 0 \\ -d & 0 & d^2-R^2t-r^2 \end{bmatrix} \;=\; (1+t)(-r^2+d^2t-r^2t-R^2t-R^2t^2).$$

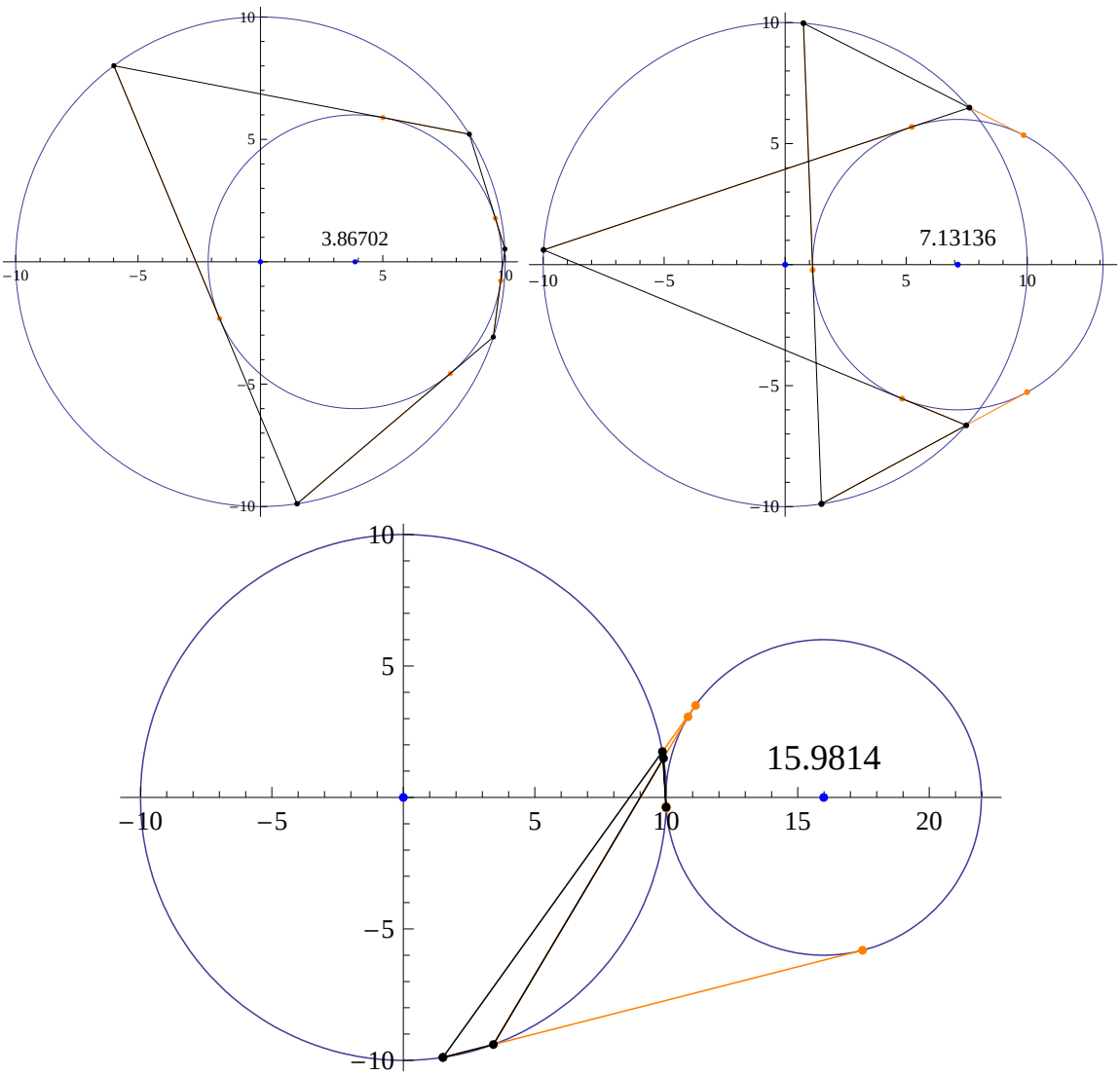
Put $R=10$, $r=6$ and $n=5$. Then the A matrix determinant for $n=5$ is a constant times

$$(-760000+56880d^2-420d^4+d^6)(1640000-25680d^2-180d^4+d^6),$$

the positive roots of which are $d=d_1$, d_2, d_3, where

$$d_1 \;\approx\; 3.8670221737, \quad d_2 \;\approx\; 7.1313558610, \quad d_3 \;\approx\; 15.9813966373.$$

When $d=d_1$ the Jacobi sc condition (1) is satisfied, but not when $d=d_2$ or d_3.



Example 8: Two ellipses and a 13-gon: Let

$$\mathcal{C}[x,y,z] \;=\; x^2+2y^2-100z^2, \quad \mathcal{D}[x,y,z] \;=\; 2xy+2y^2-36z^2+(x-dz)^2.$$

with real d. Then

$$\det Q = \det\begin{bmatrix} t+1 & 1 & -d \\ 1 & 2t+2 & 0 \\ -d & 0 & d^2-100t-36 \end{bmatrix} = -36-d^2-244t+2d^2t-472t^2+2d^2t^2-200t^3$$

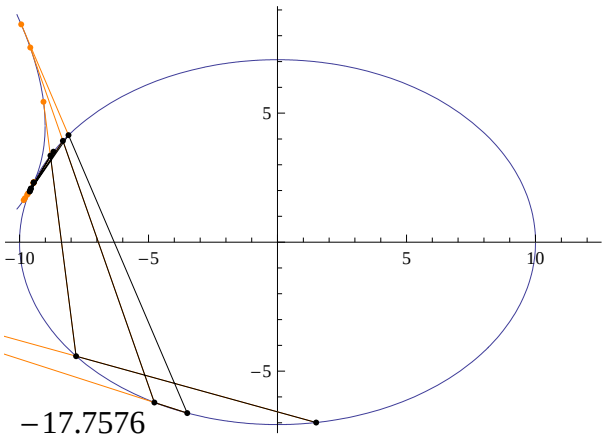
Put $n=13$. Then the A matrix determinant for $n=13$ is $P(d)/(36+d^2)^{39}$, where $P(d)$ is a monstrous polynomial of degree 84 with 16 real roots. For convenience we list the roots together with the corresponding expansions of $\sqrt{-\det(t\mathcal{C}+\mathcal{D})}$ rather than $\sqrt{\det(t\mathcal{C}+\mathcal{D})}$.

d	$\sqrt{-\det(t\mathcal{C}+\mathcal{D})}$
-17.7576	$18.7439-10.3145t-7.07045t^2+1.44431t^3-0.538756t^4+0.248343t^5-0.122212t^6$ $+0.0679406t^7-0.0355921t^8+0.0225975t^9-0.0113838t^{10}+0.00857431t^{11}-0.00363868t^{12}+O(t^{13})$
-12.6568	$14.007-2.72686t+5.14652t^2+8.14122t^3+0.639443t^4-2.86681t^5-3.159t^6$ $+0.066687t^7+2.82535t^8+2.49251t^9-0.74079t^{10}-3.3518t^{11}-2.30061t^{12}+O(t^{13})$
-6.21132	$8.636+9.6595t+17.4579t^2-7.94749t^3-8.75641t^4+25.8602t^5-14.8807t^6$ $-43.6912t^7+98.3102t^8-9.1121t^9-282.559t^{10}+425.199t^{11}+304.916t^{12}+O(t^{13})$
-1.37209	$6.15489+19.5158t+7.09756t^2-6.25752t^3+15.7489t^4-42.7204t^5+114.115t^6$ $-296.558t^7+745.144t^8-1795.38t^9+4091.72t^{10}-8595.14t^{11}+15686.6t^{12}+O(t^{13})$
-0.464898	$6.01798+20.2367t+5.15501t^2-0.717865t^3+0.206072t^4-0.0780332t^5+0.043065t^6$ $-0.0533899t^7+0.129808t^8-0.382962t^9+1.16824t^{10}-3.58253t^{11}+10.9953t^{12}+O(t^{13})$
-0.464771	$6.01797+20.2367t+5.15486t^2-0.71741t^3+0.204677t^4-0.0737517t^5+0.0299227t^6$ $-0.0130478t^7+0.00597204t^8-0.00283034t^9+0.00137706t^{10}-0.000683839t^{11}+0.000345175t^{12}+O(t^{13})$
-0.464644	$6.01796+20.2368t+5.15471t^2-0.716955t^3+0.203282t^4-0.0694701t^5+0.0167796t^6$ $+0.0272979t^7-0.117878t^8+0.377354t^9-1.16569t^{10}+3.58188t^{11}-10.9972t^{12}+O(t^{13})$
-0.383603	$6.01225+20.2674t+5.06774t^2-0.450756t^3-0.616297t^4+2.45749t^5-7.78168t^6$ $+24.1146t^7-74.579t^8+230.75t^9-714.491t^{10}+2214.13t^{11}-6866.86t^{12}+O(t^{13})$
0.383603	$6.01225+20.2674t+5.06774t^2-0.450756t^3-0.616297t^4+2.45749t^5-7.78168t^6$ $+24.1146t^7-74.579t^8+230.75t^9-714.491t^{10}+2214.13t^{11}-6866.86t^{12}+O(t^{13})$
0.464644	$6.01796+20.2368t+5.15471t^2-0.716955t^3+0.203282t^4-0.0694701t^5+0.0167796t^6$ $+0.0272979t^7-0.117878t^8+0.377354t^9-1.16569t^{10}+3.58188t^{11}-10.9972t^{12}+O(t^{13})$
0.464771	$6.01797+20.2367t+5.15486t^2-0.71741t^3+0.204677t^4-0.0737517t^5+0.0299227t^6$ $-0.0130478t^7+0.00597204t^8-0.00283034t^9+0.00137706t^{10}-0.000683839t^{11}+0.000345175t^{12}+O(t^{13})$
0.464898	$6.01798+20.2367t+5.15501t^2-0.717865t^3+0.206072t^4-0.0780332t^5+0.043065t^6$ $-0.0533899t^7+0.129808t^8-0.382962t^9+1.16824t^{10}-3.58253t^{11}+10.9953t^{12}+O(t^{13})$
1.37209	$6.15489+19.5158t+7.09756t^2-6.25752t^3+15.7489t^4-42.7204t^5+114.115t^6$ $-296.558t^7+745.144t^8-1795.38t^9+4091.72t^{10}-8595.14t^{11}+15686.6t^{12}+O(t^{13})$
6.21132	$8.636+9.6595t+17.4579t^2-7.94749t^3-8.75641t^4+25.8602t^5-14.8807t^6$ $-43.6912t^7+98.3102t^8-9.1121t^9-282.559t^{10}+425.199t^{11}+304.916t^{12}+O(t^{13})$
12.6568	$14.007-2.72686t+5.14652t^2+8.14122t^3+0.639443t^4-2.86681t^5-3.159t^6$ $+0.066687t^7+2.82535t^8+2.49251t^9-0.74079t^{10}-3.3518t^{11}-2.30061t^{12}+O(t^{13})$
17.7576	$18.7439-10.3145t-7.07045t^2+1.44431t^3-0.538756t^4+0.248343t^5-0.122212t^6$ $+0.0679406t^7-0.0355921t^8+0.0225975t^9-0.0113838t^{10}+0.00857431t^{11}-0.00363868t^{12}+O(t^{13})$

For instance, when $d \approx 1.37209$ the A matrix looks like this,

$$\begin{bmatrix} A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_7 & A_8 & A_9 & A_{10} & A_{11} & A_{12} \end{bmatrix} \approx \begin{bmatrix} 7.09756i & -6.25752i & 15.7489i & -42.7204i & 114.115i & -296.558i \\ -6.25752i & 15.7489i & -42.7204i & 114.115i & -296.558i & 745.144i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -296.558i & 745.144i & -1795.38i & 4091.72i & -8595.14i & 15686.6i \end{bmatrix},$$

and you can verify that its determinant is zero—at least approximately.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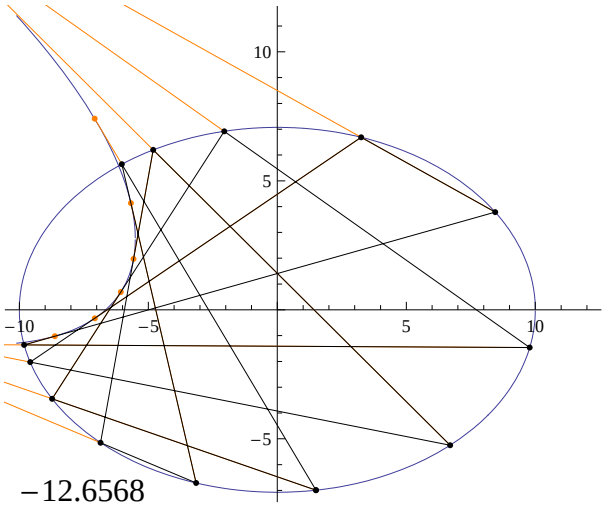
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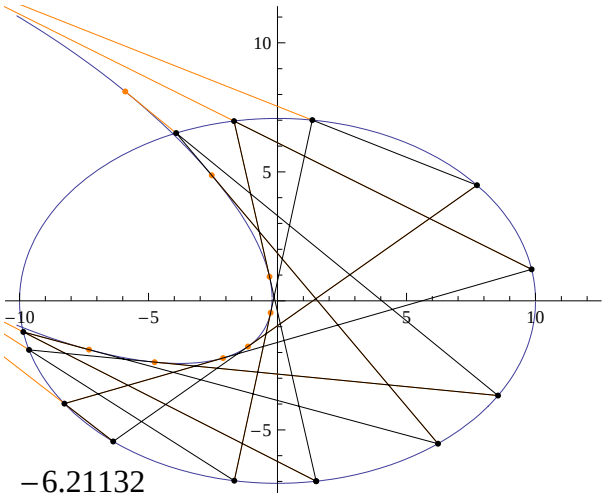
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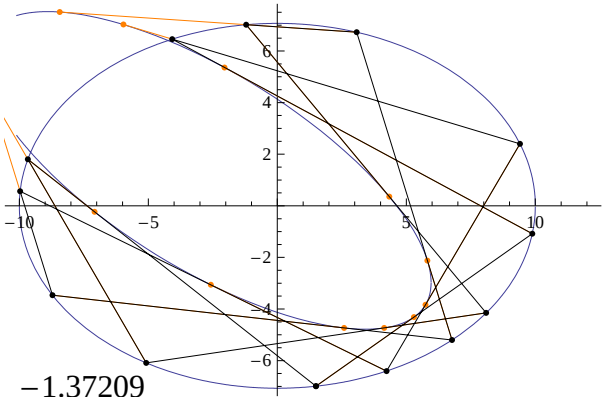
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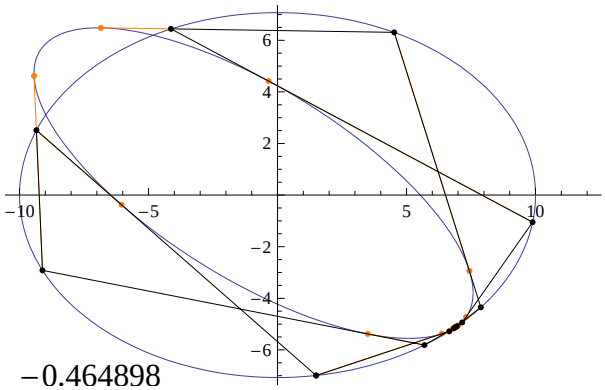
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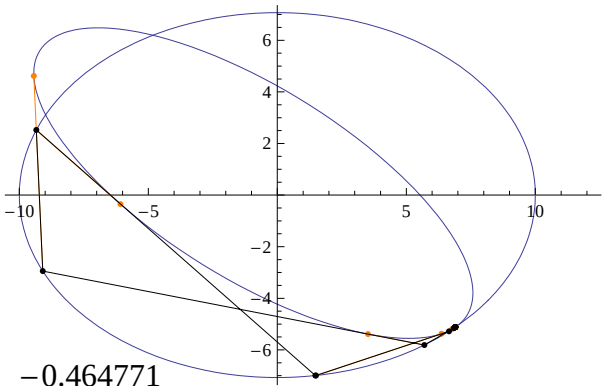
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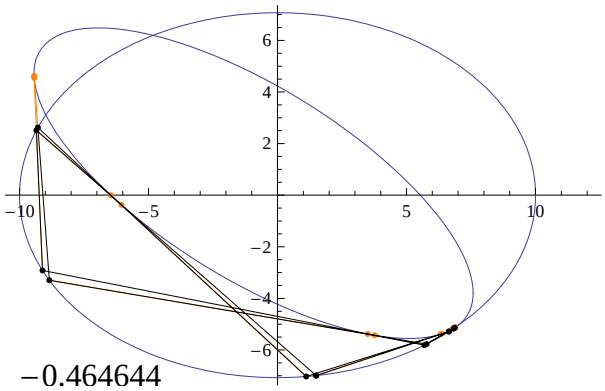
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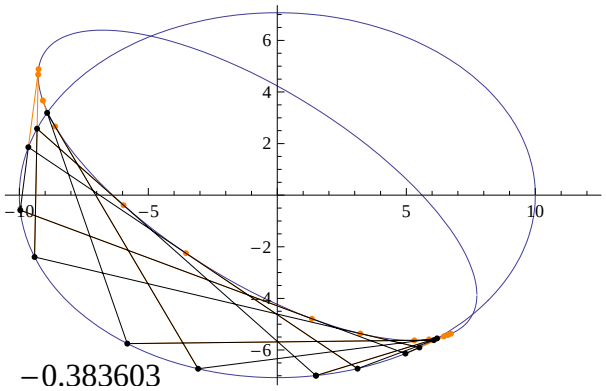
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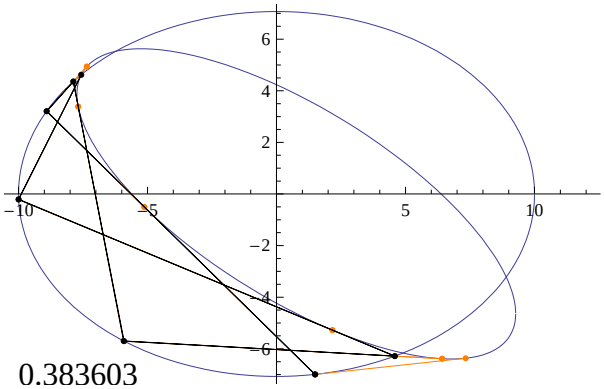
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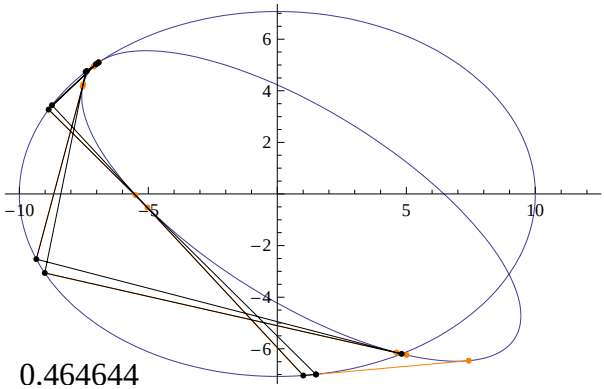
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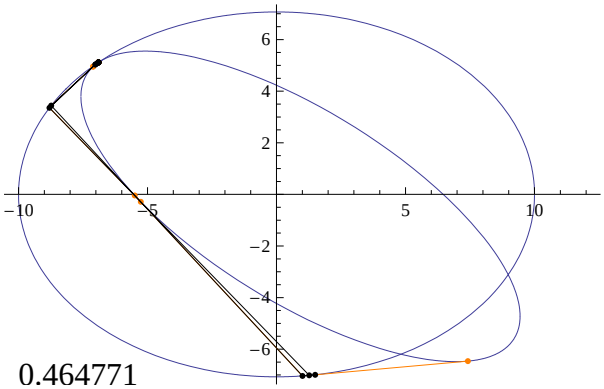
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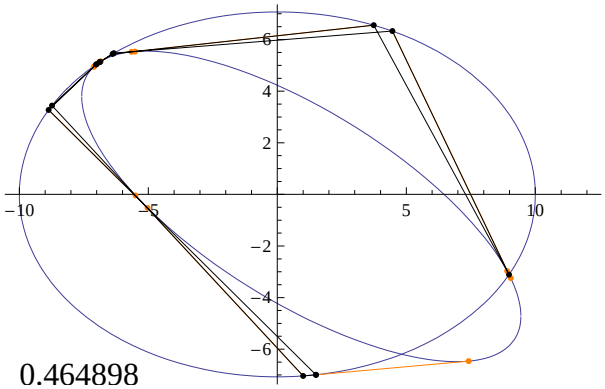
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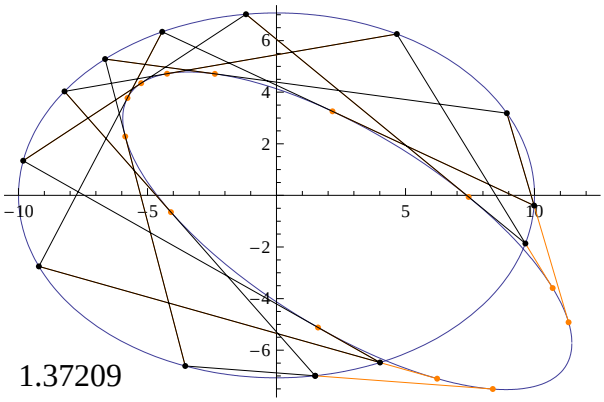
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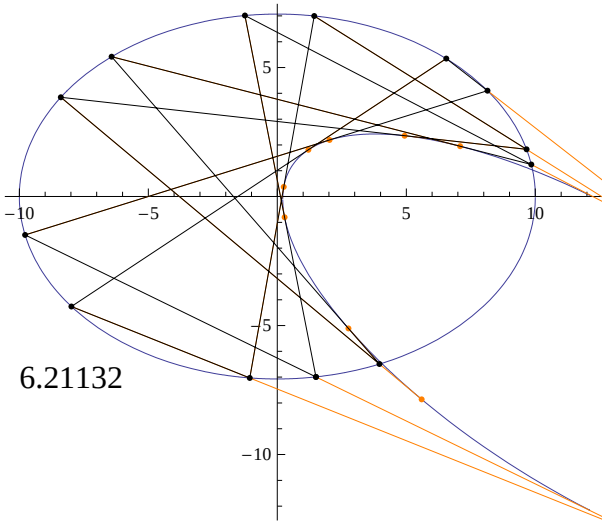
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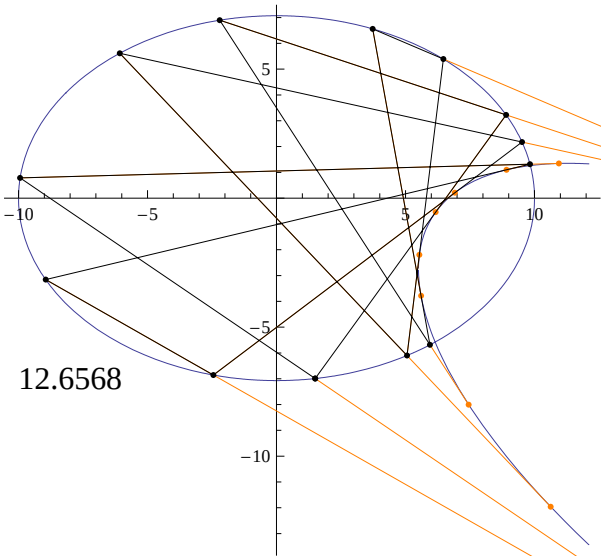
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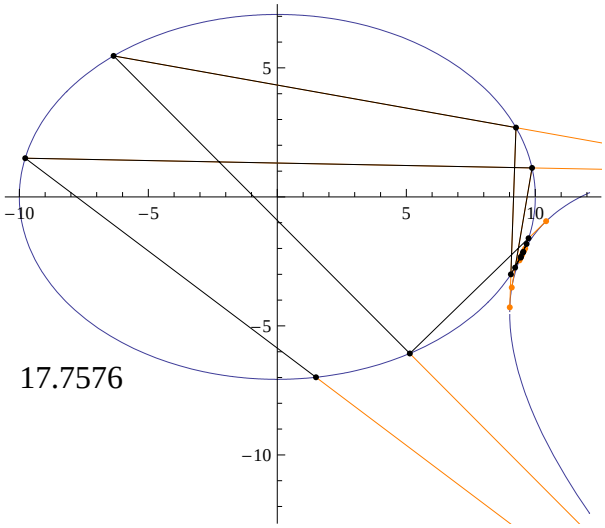

17.7576

Example 9: Two straight lines and an ellipse: Let

$$\mathcal{C}[x,y,z] \;=\; x^2-4y^2, \quad \mathcal{D}[x,y,z] \;=\; x^2+r^2y^2-z^2.$$

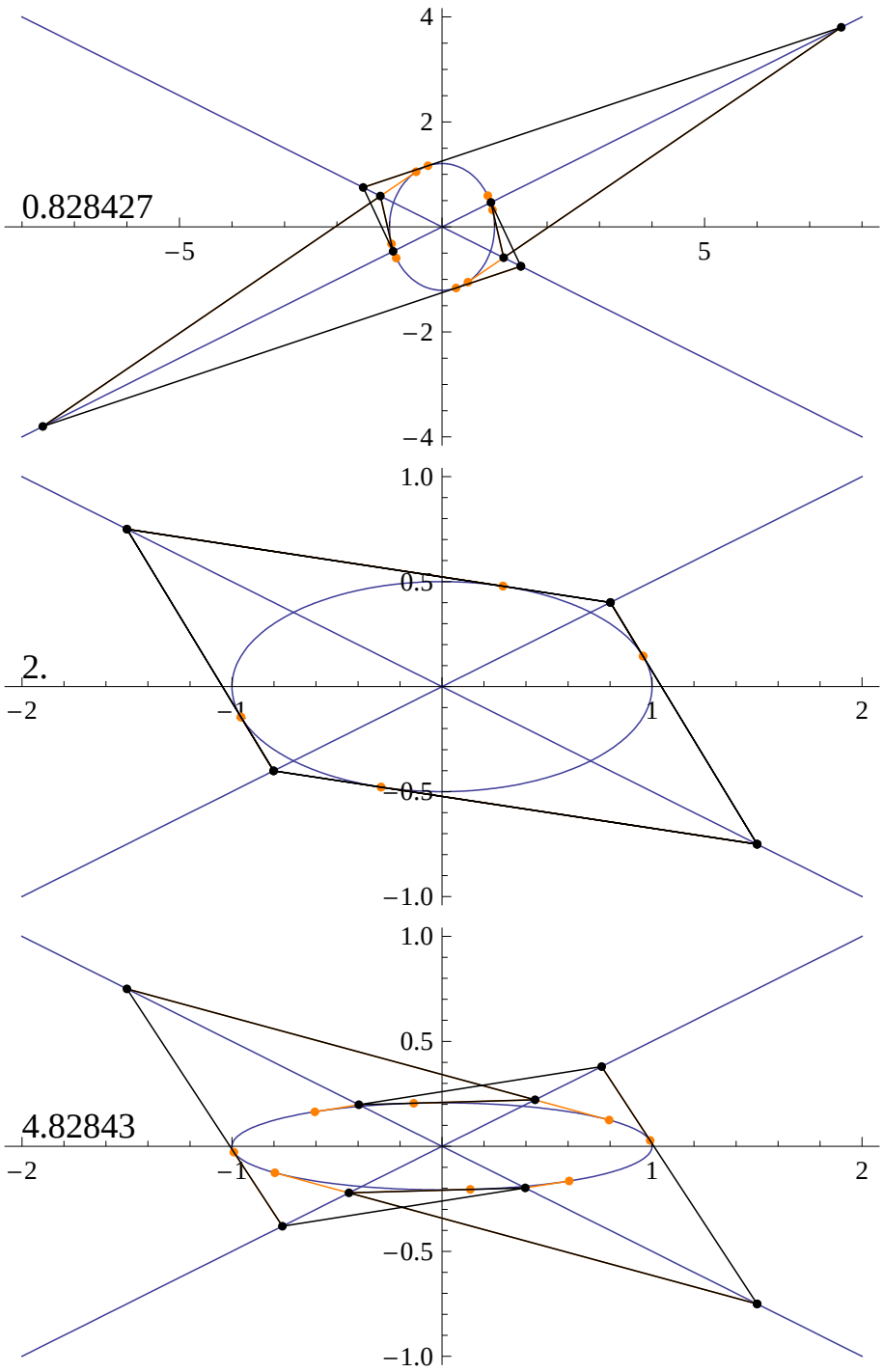
with $r>0$. Then

$$\det Q \;=\; \det\begin{bmatrix} t+1 & 0 & 0 \\ 0 & r^2-4t & 0 \\ 0 & 0 & -1 \end{bmatrix} \;=\; -r^2+4t-r^2t+4t^2$$

Put $n=8$. Then the A matrix determinant for $n=8$ is

$$\frac{-i(-2+r)(2+r)(4+r^2)^{12}(-4-4r+r^2)(-4+4r+r^2)}{33554432r^{27}},$$

which has positive roots 2 and $\sqrt{8}\pm 2$.



The connection with elliptic curves

Theorem 4 *Let $\mathcal{C}[x,y,z]=0$ and $\mathcal{D}[x,y,z]=0$ be the homogeneous quadratic equations in projective coordinates that define conics $\mathcal{C}$ and $\mathcal{D}$ and suppose that $\mathcal{E}: u^2=\det(t\mathcal{C}+\mathcal{D})$ is an elliptic curve that does not pass through the origin. Then the Cayley condition (3) for the existence of an n-gon, $n\geq 3$, circumscribed by $\mathcal{C}$ and inscribed by $\mathcal{D}$ is equivalent to a point on $\mathcal{E}$ with $t=0$ having order some factor $d\geq 3$ of n.*

Proof See [3].

We illustrate Theorem 4 with some small values of n. We may assume, by rescaling if necessary, that $\mathcal{E}$ is defined by $u^2=f(t)$ and $f(t)=\det(t\mathcal{C}+\mathcal{D})=t^3+at^2+bt+c$ with $c\neq 0$. Also we write $\phi_d(t)$ for the monic polynomial whose roots are precisely the t coordinates of points of order d on $\mathcal{E}$.

Thus when $n=3$, the Cayley condition, $A_2=0$, is equivalent to $d^2u/dt^2|_{t=0}=0$; in other words, that 0 is a point of inflexion of $\mathcal{E}$, and this is precisely the condition for $t=0$ being a point of order 3. Indeed,

$$\phi_3(t)=\frac{1}{3}\left(3t^4+4at^3+6bt^2+12ct+4ac-b^2\right) \qquad \text{and} \qquad A_2=\frac{-b^2+4ac}{8\,c^{3/2}}=\frac{3\phi_3(0)}{8\,c^{3/2}}.$$

To deal with $n=4$, we know that

$$\phi_4(t) \;=\; t^6+2at^5+5bt^4+20ct^3+(20ac-5b^2)t^2+(8a^2c-4bc-2ab^2)t+4abc-8c^2-b^3,$$

and we can perform a straightforward calculation to give

$$\left.\frac{d^3u}{dt^3}\right|_{t=0} \;=\; \left.\frac{3f'(t)^3-6f(t)f'(t)f''(t)+4f(t)^2f'''(t)}{8f(t)^{5/2}}\right|_{t=0} \;=\; \frac{b^3-4abc+8c^2}{16\,c^{5/2}} \;=\; \frac{-\phi_4(0)}{16\,c^{5/2}},$$

Hence the Cayley condition for $n=4$, $A_3=0$, is equivalent to $d^3u/dt^3|_{t=0}=0$, and this is precisely the condition that $\phi_4(0)=0$, i.e. that $t=0$ is a point of order 4.

When $n=5$ the Cayley determinant is

$$\det\begin{bmatrix}A_2 & A_3\\ A_3 & A_4\end{bmatrix} \;=\; \frac{b^6-12ab^4c+48a^2b^2c^2-32b^3c^2-64a^3c^3+128abc^3-256c^4}{1024c^5} \;=\; \frac{5\phi_5(0)}{1024c^5},$$

when $n=6$ we have

$$\begin{aligned}\det\begin{bmatrix}A_3 & A_4\\ A_4 & A_5\end{bmatrix} &= \frac{(4ac-b^2)(-3b^6+20ab^4c-16a^2b^2c^2-96b^3c^2-64a^3c^3+384abc^3-512c^4)}{16384c^7}\\ &= \frac{3\phi_3(0)\phi_6(0)}{16384c^7},\end{aligned}$$

and for $n=7$,

$$\det\begin{bmatrix}A_2 & A_3 & A_4\\ A_3 & A_4 & A_5\\ A_4 & A_5 & A_6\end{bmatrix} \;=\; \frac{7\phi_7(0)}{2097152\,c^{21/2}}.$$

References

[1] *Wikipedia*

[2] *MathWorld*

[3] P. Griffiths & J. Harris, On Cayley's explicit solution to Poncelet's porism.

Example 2 revisited: Two ellipses and an octagon

To get an elliptic curve of the form $u^2 = t^3 + \dots$ we rescale the two conics of Example 2:

$$\begin{aligned}\mathcal{C}[x,y,z] &= -\left(\frac{400}{1049}\right)^{1/3}\left(x^2+4y^2-\frac{12}{5}\,xy-\frac{1}{2}\,yz-z^2\right),\\ \mathcal{D}[x,y,z] &= -\left(\frac{400}{1049}\right)^{1/3}\left(3x^2+5y^2-r^2z^2\right).\end{aligned}$$

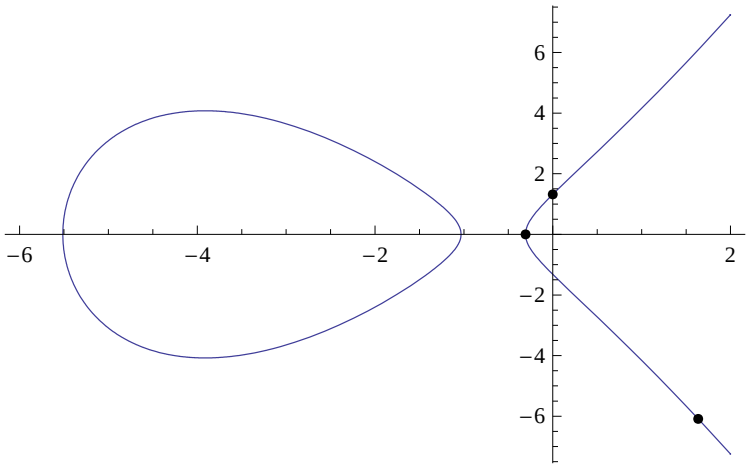
First put $r \approx\ 0.552345421970$, one of the two values we found that work for constructing an octagon circumscribed by $\mathcal{C}$ and inscribed by $\mathcal{D}$. Then

$$u^2 = \det(t\mathcal{C}+\mathcal{D}) \ = \ 1.7450074271007103+7.697408163170446t+6.851675420720262t^2+t^3$$

defines an elliptic curve. Recall the doubling formula

$$t \ \mapsto \ \frac{t^4-2bt^2-8ct-4ac+b^2}{4(t^3+at^2+bt+c)} \tag{4}$$

for the general curve $u^2 = t^3 + at^2 + bt + c$. Starting with $P_1 = (0,\ 1.3209872925583768)$ and using (4), we obtain first $2P_1 = (1.6368403744367936,\ -6.089931146512957)$ and then $4P_1 = (-0.3066641875487296,\ 0)$, at least approximately. Thus $4P_1$ is on the t-axis, confirming that it has order 2. However P_1 and $2P_1$ are not on the t-axis, and so P_1, one of the points with $t = 0$, really does have order 8, as required.



The other octagon occurs when $r \approx\ 1.0561374780$. This time the elliptic curve is

$$u^2 \ = \ 6.379941119749473 + 12.950333014839044t + 7.642704104265651t^2 + t^3.$$

Starting with $P_2 = (0,\ 2.52585453257892)$ and using the doubling formula we compute $2P_2 = (-1.0708908991221608,\ -0.21943297058394742)$, $4P_2 = (-5.498466059937016,\ 0)$, thus confirming that P_2 has order 8.