Linear recurrence sequences
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Linear recurrences

A *linear recurrence sequence* over $\mathbb{K} \subseteq \mathbb{C}$ of order r is a sequence $U_0, U_1, \ldots, \in \mathbb{K}$ defined by

$$U_n \;=\; c_{r-1}U_{n-1} + c_{r-2}U_{n-2} + \cdots + c_0 U_{n-r}, \tag{1}$$

where the $-c_i \in \mathbb{K}$ are the coefficients of the *characteristic polynomial*,

$$P(x) \;=\; x^r - c_{r-1}x^{r-1} - c_{r-2}x^{r-2} - \cdots - c_1 x - c_0.$$

We should really insist that $c_0 \neq 0$ because otherwise we can redefine (1) with a smaller value of r. Usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{Z}$ (equivalently $\mathbb{Q}$). To complete the definition of U_n we prime the recursion with r initial values,

$$U \;=\; (U_0,\ U_1,\ \ldots,\ U_{r-1}),$$

so that for $n \geq r$, $U_n = [0\ 0\ \ldots\ 1]M^{n-r+1}U$, the last element of $M^{n-r+1}U$, where M is the $r \times r$ matrix

$$M \;=\; \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ c_0 & c_1 & c_2 & \ldots & c_{r-2} & c_{r-1} \end{bmatrix}.$$

A straightforward calculation shows that $\det M = (-1)^r P(0) = (-1)^{r-1}c_0$ and that the eigenvalues of M are precisely the roots of $P(x)$ with the same multiplicities. In fact $(-1)^r P(x)$ is the characteristic polynomial of M.

Suppose $P(x)$ has t *distinct* roots $\lambda_1, \lambda_2, \ldots, \lambda_t$ and that root λ_i occurs with multiplicity t_i, $i = 1, 2, \ldots, t$, so that $r = \sum_{i=1}^{t} t_i$. Then we can attempt to construct a closed formula of the form

$$U_n \;=\; \sum_{i=1}^{t} Q_i(n)\lambda_i^n, \tag{2}$$

where $Q_i(x) = \sum_{j=0}^{t_i-1} q_{i,j}x^j$ is a polynomial of degree at most $t_i - 1$, $i = 1, 2, \ldots, t$. Observe that U_n defined by (2) satisfies the recurrence formula (1) if $n \geq r$, for

$$\sum_{h=1}^{r} c_{r-h}U_{n-h} \;=\; \sum_{h=1}^{r} c_{r-h} \sum_{i=1}^{t} \sum_{j=0}^{t_i-1} q_{i,j}(n-h)^j \lambda_i^{n-h} \;=\; \sum_{i=1}^{t} \sum_{j=0}^{t_i-1} q_{i,j} \sum_{h=1}^{r} c_{r-h}(n-h)^j \lambda_i^{n-h}$$
$$=\; \sum_{i=1}^{t} \sum_{j=0}^{t_i-1} q_{i,j} \left(n^j \lambda_i^n - \left(x\frac{d}{dx} \right)^j \left(x^{n-r}P(x) \right) \Big|_{x=\lambda_i} \right) \;=\; \sum_{i=1}^{t} \sum_{j=0}^{t_i-1} q_{i,j} n^j \lambda_i^n \;=\; U_n$$

since $d^j P(x)/dx^j = 0$ when $x = \lambda_i$ and $0 \leq j < t_i$, $i = 1, 2, \ldots, r$. To determine the polynomials we write $Q_i(x) = \sum_{j=0}^{t_i-1} q_{i,j}x^j$ and solve the linear system

$$\sum_{i=1}^{t} \sum_{j=0}^{t_i-1} n^j \lambda_i^n \, q_{i,j} \;=\; U_n, \qquad n = 0,\ 1,\ \ldots,\ r-1, \tag{3}$$

provided a solution exists in the event of the coefficient matrix of (3) being singular. For instance a singular coefficient matrix will occur if 0 is a repeated root of $P(x)$. Of course this won't happen if $c_0 \neq 0$.

If the roots are distinct, we say that the recurrence system is *simple* and in this case the polynomials $Q(x)$ are just constants.

Simple linear recurrences

Assume that the roots of $P(x)$ are distinct. Then M can be diagonalized:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \lambda_d \end{bmatrix} = R^{-1}MR,$$

where

$$R = \begin{bmatrix} \lambda_1^{-(r-1)} & \lambda_2^{-(r-1)} & \dots & \lambda_r^{-(r-1)} \\ \lambda_1^{-(r-2)} & \lambda_2^{-(r-2)} & \dots & \lambda_r^{-(r-2)} \\ & & \dots & \\ \lambda_1^{-1} & \lambda_2^{-1} & \dots & \lambda_r^{-1} \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Hence $M^n = RD^nR^{-1}$. And if we start the sequence with $U = (0, 0, \dots, 1)$, it turns out that there is a very nice closed formula for the general term [2],

$$U_n = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod\limits_{1 \leq j \leq r,\ j \neq i} (\lambda_i - \lambda_j)}, \qquad n = 0,\ 1,\ \dots.$$

Decision problems

Skolem's problem Given a linear recurrence sequence over $\mathbb{Z}$, is $U_n \neq 0$ for all $n \geq 0$? Posed by Thoralf Skolem in 1934, the problem is still open. This situation is 'faintly outrageous' (Tao) and 'a mathematical embarrassment' (Lipton).

The problem is trivial for order 1 and 'folklore' for order 2. Orders 3 and 4 were done by Vereshchagin in 1985 using results on linear logarithms by Baker and van der Poorten. In a TUCS Tech Report (2005), Havala, Harju, Hirvensalo & Karhumäaki prove 2, 3 and 4 in detail.

The ultimate positivity problem Given a linear recurrence sequence over $\mathbb{K} \subseteq \mathbb{R}$, is $U_n > 0$ for all sufficiently large n?

Order 2 over $\mathbb{R}$ done (i.e. the ultimate positivity problem is decidable for any sequence of order 2 over $\mathbb{R}$) by Burke & Webb (1981); order 3 over $\mathbb{R}$ by Nagasaka & Shiue (1990);

The positivity problem Given a linear recurrence sequence over $\mathbb{K} \subseteq \mathbb{R}$, is $U_n > 0$ for all $n \geq 0$?

Order 2 over $\mathbb{R}$ done by Havala, Harju & Hirvensalo (2006); order 3 over $\mathbb{R}$ by Laohakosol & Tangsupphathawat (2009). Orders ≤ 5 for integers were done by Ouaknine & Worrell (2014), but order 6 for integers would require 'major breakthroughs in analytic number theory'. Orders ≤ 9 for simple integer sequences were done by Ouaknine & Worrell (2014).

Linear recurrences, continued

Proposition 1 *Let U_n and V_n be linear recurrence sequence of orders r and s respectively. Then their pointwise sum $U_n + V_n$ and product U_nV_n are linear recurrence sequence of orders at most $r + s$ and rs respectively. Moreover, the order of the pointwise square U_n^2 is at most $r(r+1)/2$.*

Proof This is probably obvious.

It is clear from (2) that the ultimate fate of the sequence is determined by the *dominant roots* of $P(x)$, those roots that have maximum absolute value.

A sequence is *non-degenerate* if it is not degenerate. A sequence is *degenerate* if there are two distinct roots whose quotient is a root of 1. A degenerate sequence can be partitioned into finitely many non-degenerate sequences. For example, we can partition a real sequence with roots 3, $2i$ and $-2i$ into two real sequences each with roots 9 and -4:

$$U_n \;=\; \alpha 3^n + \beta(2i)^n + \overline{\beta}(-2i)^n; \quad U_{2n} \;=\; \alpha 9^n + 2\Re(\beta)(-4)^n, \quad U_{2n+1} \;=\; 3\alpha 9^n - 4\Im(\beta)(-4)^n.$$

Non-degeneracy implies that for any positive integer k, the number of roots of the characteristic polynomial of U_{nk} is equal to the number of roots of the characteristic polynomial of U_n. Furthermore, a simple degenerate system can be partitioned into simple non-degenerate systems. So it suffices to consider only non-degenerate linear recurrences.

So for problems concerning positivity, ultimate or otherwise of simple real linear recurrences, we can restrict ourselves to non-degenerate sequences where the dominant roots include one real positive value, and we can write (2) as

$$\frac{U_n}{\rho^n} \;=\; a + \sum_{j=1}^{m} \left(d_j \left(\frac{\lambda_j}{\rho}\right)^n + \overline{d_j}\left(\frac{\overline{\lambda_j}}{\rho}\right)^n \right) + \epsilon(n),$$

where the dominant real root is ρ, the other dominant roots if any occur in conjugate pairs $\{\lambda_j,\ \overline{\lambda_j}\}$, $j = 1, 2, \ldots, m$, $\{d_j,\ \overline{d_j}\}$, $j = 1, 2, \ldots, m$, are pairs of complex conjugate constants, a is an algebraic number, and $\epsilon(n)$ is a function that tends to zero exponentially. Note that there is no other dominant real root because otherwise it would have to be $-\rho$, and this is excluded by non-degeneracy.

Proposition 2 *Let U_n be a real linear recurrence system with no real positive dominant root. Then there are infinitely many $n \geq 0$ such that $U_n < 0$ and infinitely many $n \geq 0$ such that $U_n > 0$.*

Proof Omitted.

Theorem 1 (Skolem, Mahler, Lech, 1934-35) *Let U_n be a non-degenerate linear recurrence sequence. Then $U_n = 0$ for finitely many $n \geq 0$.*

Proof Omitted.

Proposition 3 (Mignette, 1982) *If α and β are distinct roots of a polynomial in $\mathbb{Z}[x]$ of degree d and height h, then*

$$|\alpha - \beta| \;>\; \frac{\sqrt{6}}{d^{(d+1)/2} h^{d-1}}.$$

Proof Omitted. Compare with ax^2+bx+c: $|\alpha-\beta| = |\sqrt{b^2-4ac}/a| \geq 1/h > \sqrt{3}/(2h)$.

Theorem 2 (Baker & Wüstholz, 1993) *Let $\alpha_1,\ldots,\alpha_m \in \mathbb{C}$ be algebraic numbers $\neq 0,\ 1$, and let $b_1,\ldots,b_m \in \mathbb{Z}$. Let*

$$\Lambda \;=\; b_1\log\alpha_1+\cdots+b_m\log\alpha_m, \quad \textit{where } \log z = \log|z| + i\arg z, \;\; \arg z \in (-\pi,\pi].$$

Let $A_i = \max(\textit{height of } \alpha_i,\ e)$, $i=1,\ldots,m$ and $B=\max(|b|,\ e)$. Let d be the degree of $\mathbb{Q}(\alpha_1,\ldots,\alpha_m)$ over $\mathbb{Q}$.

$$\textit{If } \Lambda\neq 0, \text{ then } \log|\Lambda| \;>\; -\,(16md)^{2(m+2}\log A_1\ldots\log A_m\log B.$$

Proof Omitted.

Corollary 1 (Ouaknine & Worrell, 2014) *There exists an integer $D\geq 1$ such that for any algebraic numbers $\lambda,\ \zeta\in\mathbb{C}$, $|\lambda|=|\zeta|=1$, for all $n\geq 2$, whenever $\lambda^n\neq\zeta$ we have*

$$|\lambda^n-\zeta| \;>\; \frac{1}{n^{(||\lambda||+||\zeta||)^D}},$$

where $||\alpha||$ is the number of bits required to store the minimum polynomial of the algebraic number α as a list of coefficients.

Proof Omitted.

Examples

Fibonacci

Recurrence: $U_n = U_{n-1}+U_{n-2}$; $P(x)=x^2-x-1$; roots: $\dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}$.
Initial values: 0, 1;

$$U_n \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n;$$

$$0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765,10946,\ldots.$$

Initial values: 1, 1; the original rabbit breeding problem;

$$U_n \;=\; \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n;$$

$$1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765,10946,\ldots.$$

Initial values: 2, 1; Lucas numbers;

$$U_n \;=\; \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n;$$

$$2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,3571,5778,9349,15127,\ldots.$$

Fibonacci with mortal rabbits

Recurrence: $U_n = U_{n-1}+U_{n-2}-U_{n-3}$; $P(x)=x^3-x^2-x+1$; roots: 1, 1, -1.
Initial values: 1, 1, 2;

$$U_n \;=\; \frac{3}{4}+\frac{n}{2}+\frac{1}{4}(-1)^n; \qquad U_{2n} \;=\; U_{2n+1} \;=\; n+1;$$

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, \ldots.$$

A simple, degenerate 4th order system

Recurrence: $U_n = 13U_{n-2} - 36U_{n-4}$; $P(x) = x^4 - 13x^2 + 36$; roots: 3, −3, 2, −2.
Initial values: 0, 0, 0, 1;

$$U_n \;=\; \frac{1}{30}3^n - \frac{1}{30}(-3)^n - \frac{1}{20}2^n + \frac{1}{20}(-2)^n; \;\; U_{2n} \;=\; 0, \;\; U_{2n+1} \;=\; \frac{1}{5}9^n - \frac{1}{5}4^n;$$

$$0, 0, 0, 1, 0, 13, 0, 133, 0, 1261, 0, 11605, 0, 105469, 0, 953317, 0, 8596237, 0, 77431669, 0, \ldots.$$

Initial values: 0, 0, 1, 1;

$$U_n \;=\; \frac{2}{15}3^n + \frac{1}{15}(-3)^n - \frac{3}{20}2^n - \frac{1}{20}(-2)^n; \;\; U_{2n} \;=\; U_{2n+1} \;=\; \frac{1}{5}9^n - \frac{1}{5}4^n;$$

$$0, 0, 1, 1, 13, 13, 133, 133, 1261, 1261, 11605, 11605, 105469, 105469, 953317, 953317, \ldots.$$

Initial values: 0, 1, 1, 1;

$$U_n \;=\; \frac{1}{5}(-3)^n + \frac{3}{10}2^n - \frac{1}{2}(-2)^n; \;\; U_{2n} \;=\; \frac{1}{5}9^n - \frac{1}{5}4^n, \;\; U_{2n+1} \;=\; -\frac{3}{5}9^n + \frac{8}{5}4^n;$$

$$0, 1, 1, 1, 13, -23, 133, -335, 1261, -3527, 11605, -33791, 105469, -312311, 953317, \ldots.$$

Initial values: 1, 1, 1, 1;

$$U_n \;=\; -\frac{2}{5}3^n - \frac{1}{5}(-3)^n + \frac{6}{5}2^n + \frac{2}{5}(-2)^n; \;\; U_{2n} \;=\; U_{2n+1} \;=\; -\frac{3}{5}9^n + \frac{8}{5}4^n;$$

$$1, 1, 1, 1, -23, -23, -335, -335, -3527, -3527, -33791, -33791, -312311, -312311, \ldots.$$

A non-simple, non-degenerate 4th order system

Recurrence: $U_n = 10U_{n-1} - 35U_{n-2} + 50U_{n-3} - 24U_{n-4}$;
$P(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$; roots: 2, 2, 1, 1.
Initial values: 0, 0, 0, 1;

$$U_n \;=\; \left(\frac{n}{2} - 2\right) 2^n + n + 2;$$

$$0, 0, 0, 1, 6, 23, 72, 201, 522, 1291, 3084, 7181, 16398, 36879, 81936, 180241, 393234, \ldots.$$

Initial values: 0, 0, 1, 1;

$$U_n \;=\; \left(\frac{-3n}{2} + 7\right) 2^n - 4n + -7;$$

$$0, 0, 1, 1, -7, -43, -159, -483, -1319, -3371, -8239, -19507, -45111, -102459, \ldots.$$

Initial values: 1, 1, 1, 1; $U_n = 1$;

A polynomial 6th order system

Recurrence: $U_n = 6U_{n-1} - 15U_{n-2} + 20U_{n-3} - 15U_{n-4} + 6U_{n-5} - U_{n-6}$;
$P(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$; roots: 1, 1, 1, 1, 1, 1.
Initial values: 0, 0, 0, 0, 0, 1;

$$U_n \;=\; \frac{(n-4)(n-3)(n-2)(n-1)n}{120};$$

$$0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003, 4368, 6188, 8568, 11628, 15504, \ldots.$$

Initial values: 0, 0, 0, 0, 1, 1;

$$U_n \;=\; -\frac{(4n-21)(n-3)(n-2)(n-1)n}{120};$$

$0, 0, 0, 0, 1, 1, -9, -49, -154, -378, -798, -1518, -2673, -4433, -7007, -10647, -15652, \ldots.$

Initial values: 1, 1, 1, 1, 1, 1; $U_n = 1$;

A degenerate system with non-real roots

Recurrence: $U_n = 3U_{n-1} - 4U_{n-2} + 12U_{n-3}$; $P(x) = x^3 - 3x^2 + 4x - 12$; roots: 3, $2i$, $-2i$.
Initial values: 0, 1, 3;

$$U_n = \frac{3}{13}3^n - \frac{3+2i}{26}(2i)^n - \frac{3-2i}{26}(-2i)^n; \quad 0, 1, 3, 5, 15, 61, 183, 485, 1455, 4621, 13863, 40565, \ldots.$$

Recurrence: $U_n = 5U_{n-1} + 36U_{n-2}$; $P(x) = x^2 - 5x - 36$; roots: 9, -4.
Initial values: 0, 3; $U_n = \frac{3}{13}9^n - \frac{3}{13}(-4)^n$; $0, 3, 15, 183, 1455, 13863, 121695, \ldots.$
Initial values: 1, 5; $U_n = \frac{9}{13}9^n + \frac{4}{13}(-4)^n$; $1, 5, 61, 485, 4621, 40565, 369181, \ldots.$

Repeated root 0

Just experimenting with the idea of putting $c_0 = 0$ in (1).
Recurrence: $U_n = 2U_{n-1}$; $P(x) = x^3 - 2x^2$; roots: 2, 0, 0.
Initial values: 1, 2, 4;

$$U_n = 2^n; \quad 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

Initial values: 1, 1, 2;

$$U_n = \frac{1}{2}2^n + \frac{1}{2}0^n; \quad 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

Initial values: 1, 0, 0; $U_n = 0^n$; $1, 0, 0, 0, 0, 0, 0, \ldots.$
Initial values: 1, 1, 1; no closed formula; $1, 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$
Initial values: 1, 2, 5; no closed formula; $1, 2, 5, 10, 20, 40, 80, 160, 320, 640, 1280, \ldots.$

References

[1] J. Ouaknine & J. Worrell, On the Positivity Problem for Simple Linear Recurrence Sequences, 2013.

[2] *M500* **203**, April 2005.