PRIMALITY WITHOUT RECOURSE TO ARITHMETIC

In which we investigate some of the properties of the Riemann zeta function and discover an interesting subset of the positive integers.

Tony Forbes

Talks for London South Bank Mathematics Study Group, 10 Oct 2012, 28 Nov 2012.

The Riemann zeta function

The Riemann zeta function is defined for complex s with $\Re s > 1$ by the series

$$\zeta(s) \;=\; \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}; \qquad -\zeta'(s) \;=\; \sum_{n=1}^{\infty} \frac{\log n}{n^s}. \tag{1}$$

n	1	2	4	6	8	10	12	14	16
$\zeta(n)$	∞	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\pi^{12}}{638512875}$	$\frac{2\pi^{14}}{18243225}$	$\frac{3617\pi^{16}}{325641566250}$

The definition (1) of $\zeta(s)$ applies to the half-plane $\Re s > 1$. However, as a complex function there is the possibility of analytic continuation left of the line $\Re s = 1$. Write

$$\eta(s) \;=\; \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \tag{2}$$

Since the series has alternating signs, $\eta(s)$ converges for $\Re s > 0$. Moreover, we have

$$\zeta(s) \;=\; \frac{2^s}{2^s - 2}\,\eta(s), \tag{3}$$

which provides analytic continuation leftwards to the line $\Re s = 0$. For example, one can compute $\eta(1/2) \approx 0.604899$ and then $\zeta(1/2) = -(\sqrt{2} - 1)\eta(1/2) \approx -1.46035$. Also we see from (3) that $\zeta(s)$ has a simple pole at $s = 1$. Indeed, we have

$$\eta(1) = \log 2, \qquad \frac{2^s}{2^s - 2} \sim \frac{1}{(\log 2)(s-1)} \quad \text{as } s \to 1; \qquad \text{hence} \qquad \zeta(s) \sim \frac{1}{s-1} \quad \text{as } s \to 1.$$

With more work one obtains this Laurent expansion around the point $s = 1$:

$$\zeta(s) \;=\; \frac{1}{s-1} + \gamma - \frac{\gamma_1}{1!}(s-1) + \frac{\gamma_2}{2!}(s-1)^2 - \frac{\gamma_3}{3!}(s-1)^3 + \ldots,$$

where $\gamma \approx 0.5772156649\ldots$ and

$$\gamma_k \;=\; \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{\log^k m}{m} - \int_1^n \frac{\log^k x}{x}\,dx \right) \;=\; \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{\log^k m}{m} - \frac{\log^{k+1} n}{k+1} \right).$$

These are the *Stieltjes* gamma constants, except for γ_0, which is Euler's constant, γ.

Although the series (2) converges in the range $0 < \Re s < 1$, it does not do so absolutely. This can cause trouble if we want to manipulate (2) by doing things like term by term differentiation.

Definition of the primes

In what follows, we aim to avoid knowledge of elementary number theory. In particular we shall try not to ask questions like 'is 26 divisible by 3?', or if we must, our answer will be 'don't know'. Therefore the usual meaning of the phrase 'n is prime' is unavailable. Instead we make the following definition.

Definition 1 We say that n *is prime* if $n \in \mathbb{P}$ for every set of positive integers $\mathbb{P}$ for which

$$\zeta(s) \;=\; \prod_{p\in\mathbb{P}} \frac{p^s}{p^s-1} \qquad \text{for all } s \text{ with } \Re s > 1. \tag{4}$$

Note the wording of our definition. If there are two sets $\mathbb{P}'$ and $\mathbb{P}''$ that satisfy (4), and, say, $31 \in \mathbb{P}'$ but $31 \notin \mathbb{P}''$, then we would have to conclude that 31 is not prime.

We can see straight away that 1 cannot belong to any set $\mathbb{P}$; for otherwise (4) implies $\zeta(2) = \infty$ whereas in fact $\zeta(2) = \pi^2/6$. Therefore 1 is not prime. Now suppose 2 is not prime. Then $1, 2 \notin \mathbb{P}$ for some set $\mathbb{P}$ in (4), and therefore

$$\frac{\pi^2}{6} \;=\; \zeta(2) \;\le\; \prod_{n=3}^{\infty}\frac{n^2}{n^2-1} \;=\; \frac{3}{2},$$

a contradiction. Therefore 2 must be prime. Similarly, if 3 is not prime, then

$$1.08232 \;<\; \frac{\pi^4}{90} \;=\; \zeta(4) \;\le\; \frac{16}{15}\cdot\prod_{n=4}^{\infty}\frac{n^4}{n^4-1} \;=\; \frac{80}{81}\cdot\prod_{n=2}^{\infty}\frac{n^4}{n^4-1} \;=\; \frac{80}{81}\cdot\frac{4\,\pi}{\sinh\pi} \;<\; 1.07469,$$

a contradiction. Therefore 3 must also be prime. However, if 4 is not prime, then

$$1.08232 \;<\; \frac{\pi^4}{90} \;=\; \zeta(4) \;\le\; \frac{16}{15}\cdot\frac{81}{80}\cdot\prod_{n=5}^{\infty}\frac{n^4}{n^4-1} \;=\; \frac{255}{256}\cdot\prod_{n=2}^{\infty}\frac{n^4}{n^4-1} \;=\; \frac{255}{256}\cdot\frac{4\,\pi}{\sinh\pi} \;<\; 1.08387$$

from which we are unable to make any useful deduction. On the other hand, if we assume that 4 is prime, then $4 \in \mathbb{P}$ for every set $\mathbb{P}$ satisfying (4) and hence

$$1.08233 \;>\; \frac{\pi^4}{90} \;=\; \zeta(4) \;\ge\; \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{4^4}{4^4-1} \;=\; \frac{2304}{2125} \;>\; 1.08423,$$

and this *is* a contradiction. Hence 4 cannot belong to any set $\mathbb{P}$ satisfying (4) and so 4 is not prime. Finally for now, if we assume 5 is not prime, then

$$1.01734 \;<\; \frac{\pi^6}{945} \;=\; \zeta(6) \;\le\; \frac{64}{63}\cdot\frac{729}{728}\cdot\prod_{n=6}^{\infty}\frac{n^6}{n^6-1} \;=\; \frac{4^6-1}{4^6}\cdot\frac{5^6-1}{5^6}\cdot\prod_{n=2}^{\infty}\frac{n^6}{n^6-1}$$

$$=\; \frac{1599507}{1600000}\cdot\frac{6\pi^2}{\cosh^2\left(\sqrt{3}\pi/2\right)} \;<\; 1.01731,$$

and this contradiction shows that 5 must be prime. We could go on to investigate 6, 7, 8 and so on. However, to deal with primes ≥ 6 it is convenient to work with the slightly more complicated function $-\zeta'(s)/\zeta(s)$ rather than $\zeta(s)$. By the way, elementary formulæ can be derived for $\prod_{n=2}^{\infty}(1-n^{-s})$, $s = 3$, 4, 6, 8, 10 and 16, using Euler's sine formula (10); see *M500* **116** (November 1989). However, I do not know of a nice expression for $\prod_{n=2}^{\infty}(1-n^{-5})$.

For $s \in \mathbb{C}$ with $\Re s > 1$ and q a positive integer, let us define

$$Q_q(s) \;=\; \frac{\log q}{q^s - 1}, \qquad S_q(s) \;=\; \sum_{p<q,\ p \text{ prime}} Q_p(s), \qquad T_q(s) \;=\; \sum_{n=q+1}^{\infty} Q_n(s),$$

assuming that we know which positive integers less than q are prime. Let us write

$$\mathcal{Z}(s) \;=\; -\frac{\zeta'(s)}{\zeta(s)}.$$

Take the logarithm of the product (4) and differentiate to get

$$-\frac{\zeta'(s)}{\zeta(s)} \;=\; \sum_{p\in\mathbb{P}} \frac{\log p}{p^s - 1}, \qquad \text{or} \qquad \mathcal{Z}(s) \;=\; \sum_{p\in\mathbb{P}} Q_p(s). \tag{5}$$

Theorem 1 *Integers 2, 3 and 5 are prime. Integers 1 and 4 are not prime. For $q \geq 6$,*

$$\begin{aligned} &\text{if } S_q(q) + T_q(q) < \mathcal{Z}(q), \text{ then } q \in \mathbb{P} \text{ for any } \mathbb{P} \text{ satisfying (4) (hence } q \text{ is prime)},\\ &\text{if } S_q(q) + Q_q(q) > \mathcal{Z}(q), \text{ then } q \notin \mathbb{P} \text{ for any } \mathbb{P} \text{ satisfying (4) (hence } q \text{ is not prime)}. \end{aligned} \tag{6}$$

As a consequence, the set $\mathbb{P}$ in (4) and (5) is unique and the problem of determining whether or not $q \geq 6$ is prime amounts to deciding which of the inequalities (6) is satisfied.

Lemma 1 *Suppose $q \geq 6$. Then*

$$Q_q(q) \;>\; T_q(q); \qquad \textit{i.e.} \quad \frac{\log q}{q^s - 1} \;>\; \sum_{n=q+1}^{\infty} \frac{\log n}{n^s - 1}.$$

Proof Exercise for reader. Although easy to verify with MATHEMATICA, it seems quite tricky to concoct a proof by hand. On the other hand, it is not too difficult to prove that $Q_q(q) \sim (e-1)T_q(q)$. □

Proof of Theorem 1 We assume $q \geq 6$ since we have already established Theorem 1 for $q < 6$. We show that exactly one of the inequalities (6) is true. From Lemma 1 we can see immediately that if one of the inequalities in (6) is false, the other must be true. Suppose on the other hand that both inequalities hold. Then $S_q(q) + T_q(q) < \mathcal{Z}(q) < S_q(q) + Q_q(q)$, and hence

$$T_q(q) \;<\; \mathcal{Z}(q) - S_q(q) \;=\; \sum_{n=q}^{\infty} \epsilon_n\, Q_n(q) \;<\; Q_q(q), \tag{7}$$

where $\epsilon_n = 0$ or 1. Of course, $\epsilon(n) = 1$ precisely when n is prime, as in the definition of $\mathcal{Z}(q)$ derived from (5). However, we do not know the meaning of the phrase 'n is prime' when $n \geq q$. All we can be sure of is that for $n \geq q$, $Q_n(q)$ might or might not belong to the sum. If $\epsilon_q = 1$, the second inequality in (7) cannot hold. But if $\epsilon_q = 0$, the first inequality in (7) becomes

$$T_q(q) \;<\; \sum_{n=q+1}^{\infty} \epsilon_n\, Q_n(q) \;\leq\; T_q(q)$$

by Lemma 1, a contradiction.

Thus we see that for $q \geq 6$, exactly one of the inequalities (6) holds. □

Properties of the primes

Theorem 2 *The are infinitely many primes.*

Proof Put $s = 2$ in the product formula:

$$\prod_{p \text{ prime}} \frac{p^2}{p^2 - 1} \;=\; \zeta(2) \;=\; \frac{\pi^2}{6}.$$

But π^2 is irrational; so there cannot be a finite number of factors in the product. $\square$

Another proof Put $s = 1 + \delta$, $\delta > 0$, in the product formula:

$$\zeta(1+\delta) \;=\; \prod_{p \text{ prime}} \frac{p^{1+\delta}}{p^{1+\delta} - 1} \;<\; \prod_{p \text{ prime}} \frac{p}{p-1} \;=\; O(1) \qquad \text{as } \delta \to 0.$$

On the other hand, using the Laurent expansion, $\zeta(s) = 1/(s-1) + \gamma + \dots$, we see that $\zeta(1+\delta) = 1/\delta + O(1) \to \infty$ as $\delta \to 0$. $\square$

Theorem 3 (The Fundamental Theorem of Arithmetic) *If n is a positive integer, then n has a unique representation $\prod_{p \text{ prime}} p^{\alpha_p}$ with non-negative integers α_p of which at most a finite number are positive.*

Proof Suppose for simplicity $\Re s > 2$. For large positive X, split the product formula for $\zeta(s)$ into two parts:

$$\begin{aligned}
\zeta(s) \;&=\; \left(\prod_{p<X,\ p \text{ prime}} \frac{1}{1 - 1/p^s}\right)\left(\prod_{p \geq X,\ p \text{ prime}} \frac{1}{1 - 1/p^s}\right) \\
&=\; \left(\prod_{p<X,\ p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right)\right)\left(\prod_{p \geq X,\ p \text{ prime}} (1 - 1/p^s)\right)^{-1} \\
&=\; \left(\sum_{n = \prod_{p \leq X,\ p \text{ prime}} p^{\alpha_p}} \frac{1}{n^s}\right) \frac{1}{1 + O(1/X)} \qquad\qquad (\text{since } \Re s > 2) \\
&\to\; \sum_{n = \prod_{p \text{ prime}} p^{\alpha_p}} \frac{1}{n^s} \qquad\qquad \text{as } X \to \infty,
\end{aligned}$$

where the α_p are non-negative integers of which at most a finite number are positive. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \zeta(s) \;\to\; \sum_{n = \prod_{p \text{ prime}} p^{\alpha_p}} \frac{1}{n^s} \tag{8}$$

and the theorem follows on equating coefficients of $1/n^s$ in (8). $\square$

The number 1 itself is the product $\prod_{p \text{ prime}} p^0$, where each exponent is zero. It is instructive (I think) to expand the product assuming, say, that 6 is prime but 7 is not:

$$\zeta(s) \;=\; \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \dots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \dots\right)\left(1 + \frac{1}{6^s} + \frac{1}{6^{2s}} + \dots\right)\dots$$

$$=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s 3^s} + \frac{1}{6^s} + \frac{1}{2^{3s}} + \dots \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{2}{6^s} + \frac{1}{8^s} + \dots.$$

Clearly something has gone wrong—the coefficient of 6^{-s} is too big and the coefficient of 7^{-s} is too small.

Properties of the Riemann zeta function

We shall compute the values of $\zeta(s)$ presented in the table on page 1. The function $x/(e^x - 1)$ is analytic near $x = 0$ and therefore has a Taylor expansion about $x = 0$:

$$\frac{x}{e^x - 1} \;=\; \sum_{n=0}^{\infty} B_n \, \frac{x^n}{n!}. \tag{9}$$

The coefficients in (9) are known as the *Bernoulli numbers*. Using l'Hôpital's rule we see that $B_0 = \lim_{x\to 0} x/(e^x - 1) = 1$. Also observe that $x/(e^x - 1) + x/2$ is an even function of x, and therefore the coefficients of odd powers of x in its Taylor expansion will be zero. Thus $B_1 = -1/2$ and $B_n = 0$ for odd $n \geq 3$. For even n we have the recursion

$$B_n \;=\; 0^n - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}.$$

n	0	1	2	4	6	8	10	12	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$

Theorem 4 ([2, Theorem 1.4]) *For non-negative integer n, we have*

$$\zeta(2n) \;=\; \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}.$$

Proof Start with Euler's sine formula

$$\sin x \;=\; x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right), \tag{10}$$

take logs and differentiate to get

$$\log \sin x \;=\; \log x + \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2\pi^2}\right), \qquad \cot x \;=\; \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2},$$

and put $x = iu/2$:

$$\frac{1}{e^u - 1} \;=\; \frac{1}{u} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2}.$$

Using (9) this gives

$$\sum_{n=1}^{\infty} B_{2n} \frac{u^{2n}}{(2n)!} \;=\; \frac{u}{e^u - 1} - 1 + \frac{u}{2} \;=\; 2\sum_{n=1}^{\infty} \frac{u^2}{4n^2\pi^2 + u^2} \;=\; 2\sum_{n=1}^{\infty} \frac{u^2}{(2n\pi)^2} \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k}}{(2n\pi)^{2k}}.$$

Reversing the order of summation yields

$$\sum_{n=1}^{\infty} B_{2n} \frac{u^{2n}}{(2n)!} \;=\; 2\sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+2}}{(2\pi)^{2k+2}} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \;=\; 2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\zeta(2k)}{(2\pi)^{2k}} \, u^{2k},$$

and the theorem follows on equating coefficients. □

Theorem 5 *The function $\zeta(s)$ defined by (1) for $\Re s > 1$ has analytic continuation over the whole complex plane except for $s = 1$, where there is a simple pole with residue* 1. *Also $\zeta(s)$ and satisfies the functional equation*

$$\frac{\Gamma((1-s)/2)}{\pi^{(1-s)/2}}\,\zeta(1-s) \;=\; \frac{\Gamma(s/2)}{\pi^{s/2}}\,\zeta(s). \tag{11}$$

Moreover $\zeta(-2n) = 0$ for $n = 1,\ 2,\ \ldots,\ \zeta(s) \neq 0$ for all other real s, and the only non-real zeros of $\zeta(s)$ occur in the strip $0 \le \Re s \le 1$.

Proof Let

$$h(x) \;=\; \left\{ \begin{array}{ll} 0 & \text{if } x \text{ is an integer,} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise} \end{array} \right\} \;=\; -\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}.$$

Then, assuming $\Re s > 1$ and observing that $\zeta(s) = s\int_1^\infty \lfloor x \rfloor x^{-s-1} dx$,

$$\zeta(s) \;=\; -\,s\int_1^\infty h(x)\frac{dx}{x^{s+1}} + \frac{1}{s-1} + \frac{1}{2}.$$

However, the right-hand side makes sense when $\Re s > -1$, and therefore provides analytic continuation of $\zeta(s)$ for $\Re s > -1$, $s \neq 1$. If $-1 < \Re s < 0$, we can extend the range of integration down to zero and get this nice formula

$$\zeta(s) \;=\; -\,s\int_0^\infty h(x)\frac{dx}{x^{s+1}} \;=\; \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty \sin(2n\pi x)\frac{dx}{x^{s+1}} \;=\; \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{(2n\pi)^s}{n}\int_0^\infty \frac{(\sin y)dy}{y^{s+1}}$$

$$=\; -\,\frac{s(2\pi)^s}{\pi}\Gamma(-s)\sin(\pi s/2)\zeta(1-s),$$

and (11) follows by application of the well known properties of the Gamma function:

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s}, \qquad \Gamma(s)\Gamma(1-s) \;=\; \frac{\pi}{\sin \pi s}, \qquad \Gamma(s)\Gamma(s+\tfrac{1}{2}) \;=\; \frac{2\sqrt{\pi}}{2^{2s}}\,\Gamma(2s).$$

The functional equation provides analytic continuation to the rest of the complex plane. The negative real zeros of $\zeta(s)$ occur because of the poles of $\Gamma(s/2)$ on the right of (11). From the product formula (4), $\zeta(s) \neq 0$ for $\Re s > 1$, and the absence of zeros in the half-plane $\Re s < 0$, except at the negative integers, follows from (11) and the properties of $\Gamma(s)$. $\square$

The *Riemann Hypothesis* states that the non-real zeros of $\zeta(s)$ only occur when $\Re s = 1/2$.

From the functional equation (11) and Theorem 4 we have a simple formula for $\zeta(n)$ when n is a negative integer. Exercise for reader: Why the sign change for $\zeta(0)$?

Theorem 6 *We have*

$$\zeta(0) \;=\; -\frac{1}{2} \qquad \textit{and} \qquad \zeta(1-n) \;=\; -\frac{B_n}{n}, \quad n = 2, 3, \ldots.$$

n	0	-1	-3	-5	-7	-9	-11	-13	-15	-17	-19	-21
B_{1-n}	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$	$\frac{854513}{138}$
$\zeta(n)$	$-\frac{1}{2}$	$-\frac{1}{12}$	$\frac{1}{120}$	$-\frac{1}{252}$	$\frac{1}{240}$	$-\frac{1}{132}$	$\frac{691}{32760}$	$-\frac{1}{12}$	$\frac{3617}{8160}$	$-\frac{43867}{14364}$	$\frac{174611}{6600}$	$-\frac{77683}{276}$

An *integral (entire) function of order* $n > 0$ is a function $f(s)$ analytic over the whole complex plane such that for any $\epsilon > 0$ but for no $\epsilon < 0$, $\log|f(s)| \ll |s|^{n+\epsilon}$ as $s \to \infty$. Multiplying the right-hand side of (11) by s and $(s-1)$ to eliminate the simple poles of $\Gamma(0)$ and $\zeta(1)$ respectively, we obtain this integral function of order 1:

$$\xi(s) \;=\; s(s-1)\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s). \tag{12}$$

J. Hadamard showed that an integral function of finite order can be written as a product, $e^{A+Bs}\prod_\rho(1-s/\rho)e^{s/\rho}$, where ρ runs through its roots. For $\xi(s)$ we see that the negative integer zeros of $\zeta(s)$ have been removed by the poles of $\Gamma(s/2)$, and no further zeros have been introduced. So the only zeros of $\xi(s)$ are the non-real zeros of $\zeta(s)$ in the strip $0 < \Re s < 1$. Therefore

$$\xi(s) \;=\; e^{A+Bs}\prod_\rho(1-s/\rho)e^{s/\rho}, \qquad \text{or} \qquad \frac{\xi'(s)}{\xi(s)} \;=\; B + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{13}$$

for some constants A and B and where ρ ranges over the non-real zeros of $\zeta(s)$. In fact $A = 0$ and $B = \log 2\sqrt{\pi} - 1 - \frac{1}{2}\gamma$, as can be seen by expanding $\xi(s)$ as a Taylor series about 0. Recalling the definition of $\mathcal{Z}(s)$, (2) and (13) yield this interesting formula:

$$\mathcal{Z}(s) \;=\; -\frac{\zeta'(s)}{\zeta(s)} \;=\; 1 - \log 2\pi + \frac{\gamma}{2} + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} - \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{14}$$

From the functional equation (11) and the fact that $\zeta(\overline{s}) = \overline{\zeta(s)}$, we see that the non-real zeros of $\zeta(s)$ are symmetrically placed about the real axis and the line $\Re s = 1/2$. Thus $\zeta(\frac{1}{2}+\delta+it) = 0$ implies $\zeta(\frac{1}{2}+\delta-it) = \zeta(\frac{1}{2}-\delta+it) = \zeta(\frac{1}{2}-\delta-it) = 0$.

Theorem 7 ([2, Theorem 1.7]) *Let $N(T)$ denote the number of zeros of $\zeta(s)$ in the rectangle $0 < \Re s < 1$, $0 < \Im s \le T$. Then*

$$N(T) \;=\; \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Proof Let $\mathcal{D}$ denote the rectangle with vertices $2 \pm iT$ and $-1 \pm iT$. Since the only zeros of $\zeta(s)$ inside $\mathcal{D}$ correspond to zeros of the analytic function $\xi(s)$, and since $N(T)$ counts only the upper half of them, we have

$$N(T) \;=\; \frac{1}{4\pi i}\int_{\mathcal{D}}\frac{\xi'(s)}{\xi(s)}ds.$$

The remainder of the proof consists of the lengthy estimation of the integral. The details are omitted. □

As a consequence of Theorem 7 we obtain some useful information about the distribution of the non-real zeros of the zeta function:

$$\sum_{|\tau|<T}\frac{1}{|\tau|} \ll (\log T)^2, \qquad \sum_{|\tau|<T}\frac{1}{|\tau^2|} \ll \frac{\log T}{T}, \qquad \tau_n \sim \frac{2\pi n}{\log n} \quad (n\to\infty), \tag{15}$$

where τ runs through the imaginary parts of the non-real zeros of $\zeta(s)$, and $0 < \tau_1 \le \tau_2 \le \ldots$ are the imaginary parts, arranged in order, of the non-real zeros of $\zeta(s)$ in the upper half-plane.

The *von Mangoldt function* $\Lambda(n)$ and its sum function $\psi(x)$ are defined by

$$\Lambda(p^k) = \left\{ \begin{array}{ll} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{if } n \text{ is not a prime power} \end{array} \right\}, \qquad \psi(x) = {\sum_{n\le x}}' \Lambda(n),$$

where $\sum'$ indicates that at discontinuities (i.e. when x is a prime power) we must remember to take the mean value, $\frac{1}{2}\big(\psi\big(x-\frac{1}{2}\big)+\psi\big(x+\frac{1}{2}\big)\big)$. From the product formula (5) we obtain

$$\mathcal{Z}(s) \;=\; \sum_{p \text{ prime}} \frac{\log p}{p_s - 1} \;=\; \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}} \;=\; \sum_{k=1}^{\infty} \sum_{p \text{ prime}} \frac{\log p}{(p^k)^s} \;=\; \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Theorem 8 (von Mangoldt's Formula) *For $x > 1$, we have*

$$\psi(x) \;=\; x - \sum_{\rho,\ \zeta(\rho)=0,\ \Im\rho\neq 0} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} + \mathcal{Z}(0).$$

Proof Since $\psi(x)$ is the sum function of the coefficients of the Dirichlet series $\mathcal{Z}(s)$, we have

$$\psi(x) \;=\; \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \mathcal{Z}(s)\,\frac{x^s}{s}\,ds, \qquad \text{and} \qquad \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s}\,ds \;=\; \left\{ \begin{array}{ll} 0 & \text{if } 0 \le x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{array} \right.$$

Now deform the $\psi(x)$ contour to include all the poles of the integrand. Details omitted. □

Theorem 9 *For real t, $\zeta(1+it) \neq 0$.*

Proof Assume that $1+it$ is a zero of order $m \ge 1$ and since $\zeta(s)$ has a pole at $s = 1$, we can also assume that $t \neq 0$. Then $1+it$ is a simple pole of $\mathcal{Z}(s) = -\zeta'(s)/\zeta(s)$ with residue $-m$. Moreover $\mathcal{Z}(s)$ has a simple pole with residue 1 at $s = 1$. Thus, as $\delta > 0$, $\delta \to 0$, we have

$$\mathcal{Z}(1+\delta) \;\sim\; \frac{1}{\delta}, \qquad \mathcal{Z}(1+\delta+it) \;\sim\; \frac{-m}{\delta}, \qquad \mathcal{Z}(1+\delta+2it) \;\sim\; \frac{-r}{\delta}$$

for some $r \ge 0$, and so

$$\Re\left(3\mathcal{Z}(1+\delta) + 4\mathcal{Z}(1+\delta+it) + \mathcal{Z}(1+\delta+2it)\right) \;\sim\; \frac{3-4m-r}{\delta} \;<\; 0.$$

But

$$\begin{aligned} &\Re\left(3\mathcal{Z}(1+\delta) + 4\mathcal{Z}(1+\delta+it) + \mathcal{Z}(1+\delta+2it)\right) \\ &\quad= \sum_{n=1}^{\infty}(3+4\cos(t\log n)+\cos(2t\log n))\frac{\Lambda(n)}{n^{1+\delta}} \;=\; \sum_{n=1}^{\infty} 2\big(1+\cos(t\log n)\big)^2\frac{\Lambda(n)}{n^{1+\delta}} \;\ge\; 0, \end{aligned}$$

and this contradiction proves the theorem. □

With a lot more work we can obtain a significantly wider zero-free region.

Theorem 10 *There exists positive constants C and t_0 such that $\zeta(\sigma+it) \neq 0$ in the region*

$$\sigma \;>\; 1 - \frac{C}{(\log t)^{2/3}(\log\log t)^{1/3}}, \qquad t \;\ge\; t_0.$$

Proof Omitted. See [2, Theorem 6.1].

Theorem 11 (The Prime Number Theorem) *For some positive constant C,*

$$\psi(x) - x \;=\; O\left(x\exp\left(-\frac{C(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right).$$

Proof Follows from (15) and Theorems 8 and 10. Details are left to the reader. □

Postscript

Theorem 12 **([1, Theorem 49])** *π^2 is irrational.*

Proof For positive integer n, define

$$f(x) \;=\; \frac{x^n(1-x)^n}{n!} \;=\; \frac{1}{n!}\sum_{m=n}^{2n} c_m x^m,$$

where the c_m are integers. Then

$$f(1-x) \;=\; f(x), \qquad 0 \;<\; f(x) \;<\; \frac{1}{n!} \quad \text{for } x \in (0,1), \qquad f(0) \;=\; f(1) \;=\; 0,$$

$$f^{(m)}(0) \;=\; (-1)^m f^{(m)}(1) \;=\; \begin{cases} 0 & \text{for } m < n \text{ or } m > 2n, \\ \dfrac{m!\, c_m}{n!} \;\in\; \mathbb{Z} & \text{for } n \le m \le 2n. \end{cases}$$

Suppose $\pi^2 = a/b$, where a and b are positive integers. Let

$$G(x) \;=\; b^n \sum_{m=0}^{n} (-1)^m \pi^{2n-2m} f^{(2m)}(x)$$

and observe that G(0) and G(1) are integers. Also

$$\begin{aligned} \frac{d}{dx}\left(G'(x)\sin\pi x - \pi G(x)\cos\pi x\right) \;&=\; \big(G''(x) + \pi^2 G(x)\big)\sin\pi x \\ &=\; b^n \pi^{2n+2} f(x)\sin\pi x \;=\; a^n\pi^2 f(x)\sin\pi x. \end{aligned}$$

Hence

$$\frac{1}{\pi}\int_0^1 a^n\pi^2 f(x)\sin\pi x\, dx \;=\; \left[\frac{G'(x)\sin\pi x}{\pi} - G(x)\cos\pi x\right]_0^1 \;=\; G(0) - G(1),$$

an integer. On the other hand,

$$0 \;<\; \frac{1}{\pi}\int_0^1 a^n\pi^2 f(x)\sin\pi x\, dx \;\le\; \frac{\pi a^n}{n!} \;<\; 1$$

for sufficiently large n, a contradiction. □

Theorem 13 **([1, Theorem 48])** *If h is a positive integer, e^h is irrational.*

Proof This is similar to Theorem 12 but somewhat less complicated. We use instead of $G(x)$ the function

$$F(x) \;=\; \sum_{m=0}^{2n} (-1)^m h^{2n-m} f^{(m)}(x),$$

which has the property

$$\frac{d}{dx}\left(e^{hx}F(x)\right) \;=\; e^{hx}\left(hF(x) + F'(x)\right) \;=\; h^{2n+1}e^{hx}f(x).$$

If $e^h = a/b$ and n is sufficiently large, then

$$0 \;<\; aF(1) - bF(0) \;=\; b\left[e^{hx}F(x)\right]_0^1 \;=\; b\int_0^1 h^{2n+1}e^{hx}f(x)dx \;<\; \frac{bh^{2n}e^h}{n!} \;<\; 1,$$

a contradiction. □

Lemma 2 *As $q \to \infty$, $Q_q(q) \sim (e-1)T_q(q)$.*

Proof Clearly,

$$\begin{aligned}
T_q(q) &= \sum_{n=q+1}^{\infty} Q_n(q) = \sum_{n=q+1}^{\infty} \frac{\log n}{n^q - 1} \sim \sum_{n=q+1}^{\infty} \frac{\log n}{n^q} \\
&\sim \frac{\log(q+1)}{(q+1)^q} + \frac{\log(q+2)}{(q+2)^q} + \frac{\log(q+3)}{(q+3)^q} + \ldots \sim \frac{\log q}{eq^q} + \frac{\log q}{e^2 q^q} + \frac{\log q}{e^3 q^q} + \ldots \\
&\sim \frac{\log q}{(e-1)q^q} \sim \frac{Q_q(q)}{e-1}.
\end{aligned}$$

$\square$

Proof of Lemma 1 Let

$$\frac{T_q(q)}{Q_q(q)} = \frac{1}{Q_q(q)} \sum_{n=q+1}^{\infty} Q_n(q) = \sum_{m=1}^{\infty} \frac{\log(q+m)}{(q+m)^q - 1} \cdot \frac{q^q - 1}{\log q} = \sum_{m=1}^{\infty} K_m(q) L_m(q),$$

say, where

$$K_m(q) = \frac{q^q - 1}{(q+m)^q - 1}, \quad \text{and} \quad L_m(q) = \frac{\log(q+m)}{\log q}.$$

Then

$$\frac{L_m(q+1)}{L_m(q)} = \frac{\log(q+m+1)}{\log(q+m)} \cdot \frac{\log q}{\log(q+1)},$$

which is clearly less than 1 when $m \geq 1$ and $q \geq 2$. And with a bit more work we see that

$$\frac{K_m(q+1)}{K_m(q)} = \frac{(q+1)^{q+1} - 1}{(q+m+1)^{q+1} - 1} \cdot \frac{(q+m)^q - 1}{q^q - 1}$$

is also less than 1 when $m \geq 1$ and $q \geq 6$. So $T_q(q)/Q_q(q)$ is decreasing for $q \geq 6$, and the proof of the lemma is completed by observing that $T_6(6)/Q_6(6) < 0.91$. $\square$

Infinite products

Let

$$f(s) = \prod_{n=2}^{\infty} \left(1 - n^{-s}\right), \quad g(s) = \prod_{n=2}^{\infty} \left(1 + n^{-s}\right).$$

Then

$$f(2) = \frac{1}{2}, \quad g(2) = \frac{\sinh \pi}{2\pi}, \quad f(3) = \frac{\cosh(\sqrt{3}\pi/2)}{3\pi}, \quad g(3) = \frac{\cosh(\sqrt{3}\pi/2)}{2\pi},$$

$$f(4) = f(2)g(2) = \frac{\sinh \pi}{4\pi}, \quad g(4) = \frac{\cosh \sqrt{2}\pi - \cos \sqrt{2}\pi}{4\pi^2}, \quad f(6) = f(3)g(3) = \frac{\cosh^2(\sqrt{3}\pi/2)}{6\pi^2},$$

$$f(8) = f(4)g(4) = \frac{(\sinh \pi)(\cosh \sqrt{2}\pi - \cos \sqrt{2}\pi)}{16\pi^3},$$

$$f(10) = \frac{(\sinh^2 B_- + \sin^2 A)(\sinh^2 B_+ + \cos^2 A)}{10\pi^4}, \quad A = \frac{\pi(\sqrt{5}+1)}{4}, \quad B_\pm = \frac{\pi}{2}\sqrt{\frac{5 \pm \sqrt{5}}{2}}.$$

References

[1] G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers.*

[2] A. Ivić, *The Riemann Zeta-Function.*