The Four Colour Theorem at Fifty

Maps, matroids and the proof

Carrie Rutherford

LSBU Maths Study Group

www.theoremoftheday.org/MathsStudyGroup/

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Summer and autumn, 1976

- **June 1976.** Appel and Haken announce a proof of the Four Colour Conjecture at the University of Illinois.
- **5 August 1976.** A family departs Kotoka International Airport, Accra, Ghana – as recorded by the immigration stamp shown here.
- **Autumn 1976.** First term of secondary school in England. A geography lesson. The teacher states, as plain fact:

"Any map can be coloured with four colours, so that neighbouring countries never share a colour."

- No proof offered. None asked for. The theorem was months old. Perhaps the teacher knew.

Two months after the proof.

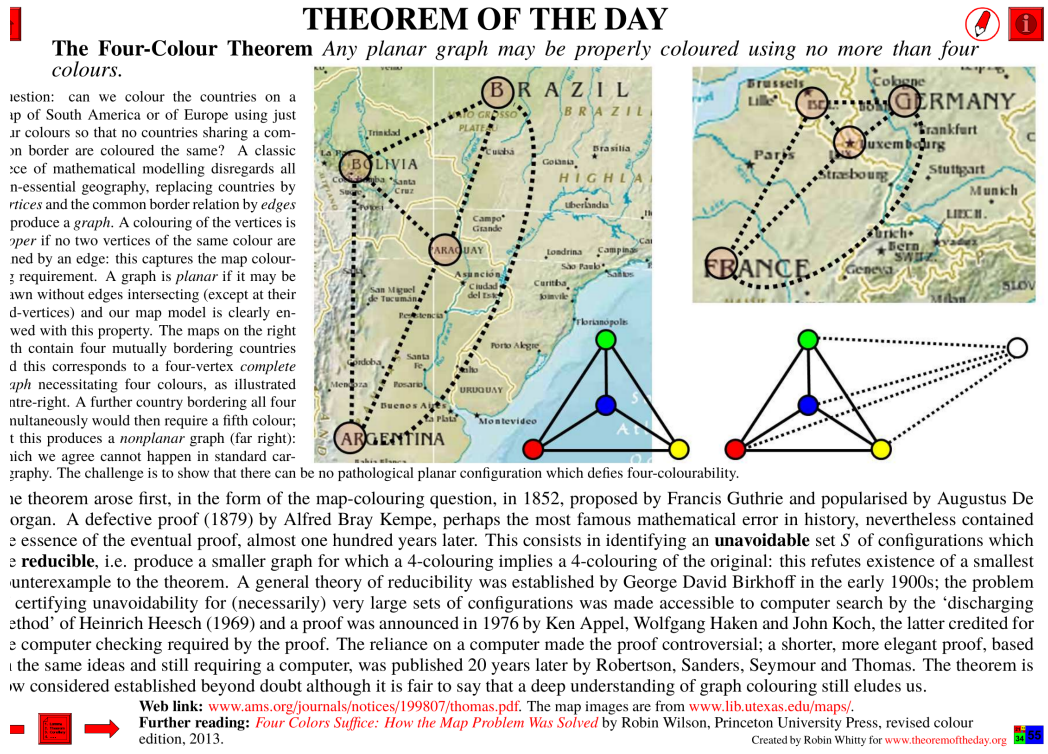
Plan: the history is the structure

1852	Guthrie asks the question
1879–1890	Kempe proves it; Heawood unproves it
1912	Birkhoff makes it a polynomial
1932–1935	Whitney, duality, and the birth of matroids
1947–1954	Tutte: flows are dual colourings
1969–1976	Heesch, discharging, Appel and Haken
1997–2005	The proof is cleaned, then formalised

- Each stage left behind a real theorem.
- Where the proof is short enough to give in full, we do; otherwise we sketch.

Theorem of the Day #1 – 16 November 2005

Robin Whitty's *Theorem of the Day* [25] launched on 16 November 2005 with the Four Colour Theorem as its very first entry – shortly after Robin and I became colleagues at LSBU. It has since grown to over 400 theorems.



THEOREM OF THE DAY

The Four-Colour Theorem *Any planar graph may be properly coloured using no more than four colours.*

uestion: can we colour the countries on a ap of South America or of Europe using just ar colours so that no countries sharing a com- on border are coloured the same? A classic ece of mathematical modelling disregards all n-essential geography, replacing countries by *rtices* and the common border relation by *edges* produce a *graph*. A colouring of the vertices is *oper* if no two vertices of the same colour are ned by an edge: this captures the map colour- requirement. A graph is *planar* if it may be awn without edges intersecting (except at their d-vertices) and our map model is clearly en- wed with this property. The maps on the right th contain four mutually bordering countries d this corresponds to a four-vertex *complete aph* necessitating four colours, as illustrated ntre-right. A further country bordering all four nultaneously would then require a fifth colour; t this produces a *nonplanar* graph (far right): hich we agree cannot happen in standard car- graphy. The challenge is to show that there can be no pathological planar configuration which defies four-colourability.

BRAZIL
BOLIVIA
ARGENTINA
PARAGUAY
URUGUAY
GERMANY
FRANCE

he theorem arose first, in the form of the map-colouring question, in 1852, proposed by Francis Guthrie and popularised by Augustus De organ. A defective proof (1879) by Alfred Bray Kempe, perhaps the most famous mathematical error in history, nevertheless contained e essence of the eventual proof, almost one hundred years later. This consists in identifying an **unavoidable** set S of configurations which e **reducible**, i.e. produce a smaller graph for which a 4-colouring implies a 4-colouring of the original: this refutes existence of a smallest unterexample to the theorem. A general theory of reducibility was established by George David Birkhoff in the early 1900s; the problem certifying unavoidability for (necessarily) very large sets of configurations was made accessible to computer search by the 'discharging ethod' of Heinrich Heesch (1969) and a proof was announced in 1976 by Ken Appel, Wolfgang Haken and John Koch, the latter credited for e computer checking required by the proof. The reliance on a computer made the proof controversial; a shorter, more elegant proof, based the same ideas and still requiring a computer, was published 20 years later by Robertson, Sanders, Seymour and Thomas. The theorem is w considered established beyond doubt although it is fair to say that a deep understanding of graph colouring still eludes us.

Web link: www.ams.org/journals/notices/199807/thomas.pdf. The map images are from www.lib.utexas.edu/maps/.
Further reading: *Four Colors Suffice: How the Map Problem Was Solved* by Robin Wilson, Princeton University Press, revised colour edition, 2013.
Created by Robin Whitty for www.theoremoftheday.org

1852: colouring the counties of England

- Francis Guthrie, colouring a map of the English counties, notices that four colours always seem to suffice.
- His brother Frederick asks their lecturer, Augustus De Morgan.
- De Morgan writes to Hamilton, 23 October 1852: he cannot prove it, and cannot find a map needing five.
- Hamilton is unmoved. The question sleeps until Cayley raises it at the London Mathematical Society in 1878.

Making the question precise

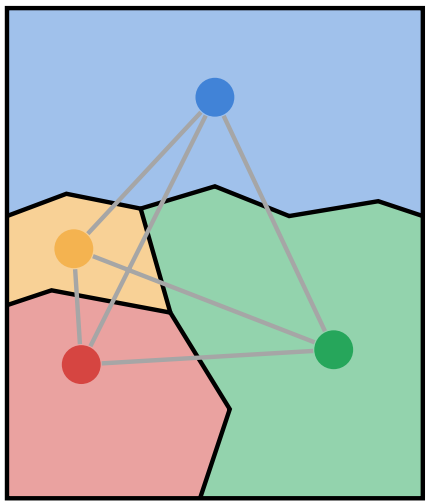
- Regions of a map $\longleftrightarrow$ vertices of the *dual* plane graph.
- Shared borders $\longleftrightarrow$ edges.
- A map colouring is a *proper vertex colouring* of a planar graph: adjacent vertices get different colours.

Chromatic number

$\chi(G)$ = minimum number of colours needed for a proper vertex colouring of G.

Conjecture (Four Colour Conjecture)

Every planar graph G satisfies $\chi(G) \leq 4$.



A map (four coloured regions) and its dual graph.

The engine: Euler's formula

Theorem (Euler, 1750s)

For a connected plane graph with v vertices, e edges and f faces,

$$v - e + f = 2.$$

Proof.

Induction on e. If G is a tree then $e = v - 1$ and $f = 1$. Otherwise pick an edge on a cycle and delete it: e drops by one, and the two faces on either side merge, so f drops by one. $v - e + f$ is unchanged. □

Two corollaries we will use constantly

Corollary

A simple planar graph with $v \geq 3$ vertices has $e \leq 3v - 6$ edges.

Proof.

Each face is bounded by at least 3 edges and each edge borders at most 2 faces, so $e \geq \frac{3f}{2}$. Substitute into Euler:
$2 = v - e + f \leq v - e + \frac{2e}{3} = v - \frac{e}{3}$, giving $e \leq 3v - 6$. □

Corollary

Every simple planar graph has a vertex of degree at most 5.

Proof.

If every degree were ≥ 6 then
$e = \frac{1}{2}\sum_v \deg(v) \geq \frac{1}{2} \cdot 6v = 3v > 3v - 6$. □

- These inequalities underpin every approach from 1879 to 1976.

1879: Kempe's proof

- Alfred Kempe, barrister and amateur mathematician, publishes a proof in the *American Journal of Mathematics* (1879). Robin Whitty (*Theorem of the Day* [25]) calls it "perhaps the most famous mathematical error in history".
- The strategy: suppose a smallest planar graph with $\chi > 4$ exists. It contains u with $\deg(u) \leq 5$. Delete u, four-colour the rest by minimality, then show u can be coloured.
- $\deg(u) \leq 3$: at least one colour is spare at u.
- $\deg(u) \in \{4, 5\}$: Kempe invents *Kempe chains*.
- The proof stands for eleven years. Kempe is elected FRS and later knighted.
- The proof is wrong – but the strategy and the chains survive into every subsequent approach.

Kempe chains and the degree 4 case

Kempe's degree 4 argument. *If* $\deg(u) = 4$*,* u *can always be recoloured* – so no smallest counterexample has a degree-4 vertex. The diagram illustrates the proof.

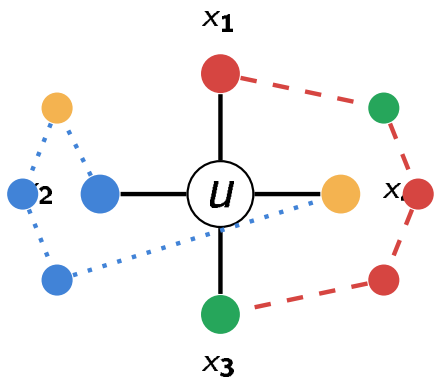
Proof.

u has neighbours x_1, x_2, x_3, x_4 with distinct colours (else spare). Goal: free a colour for u.

Case 1. x_1, x_3 in *different* red–green chains: swap red↔green on x_1's chain, so x_1 becomes green, freeing red for u.

Case 2. x_1, x_3 in the *same* red–green chain: with u this separates x_2 from x_4 (planarity), so x_2, x_4 in *different* blue–yellow chains. Swap on x_2's chain, freeing blue for u. □

An (a, b)-*chain* is a component of the subgraph on vertices coloured a or b; swapping $a \leftrightarrow b$ on one chain gives another valid colouring.



Case 2: red–green chain (—) encircles x_2, so blue–yellow chain $(\cdots)$ is split.

Kempe's degree 5 argument

With $\deg(u) = 5$ and only 4 colours, two neighbours share a colour. Say x_1(red), x_2(blue), x_3(green), x_4(yellow), x_5(blue). Kempe applies the *same case analysis as degree 4*, aiming to free red for u:

- **Case 1.** x_1, x_3 in *different* red–green chains: swap x_1's chain so x_1 becomes green, freeing red for u.
- **Case 2.** x_1, x_3 in the *same* red–green chain: by planarity, x_2 and x_4 are in *different* blue–yellow chains. Swap x_2's chain, freeing blue for u.

The cases are mutually exclusive and exhaustive – the same structure as for degree 4. Kempe's paper (1879) stood unchallenged for eleven years.

Heawood's counterexample (1890)

With 5 neighbours and only 4 colours, one colour – say blue – must appear twice (at x_2 and x_5). This is what breaks Kempe's Case 2.

- The blue–yellow chain through x_2 and the blue–green chain through x_5 *share a colour* (blue). This cannot happen for degree 4, where all four neighbour colours are distinct.
- Swapping blue↔yellow throughout x_2's chain changes some shared blue vertex to yellow, breaking the blue–green chain.
- After the swap: x_5 is still blue, and x_4 (formerly yellow, now blue after the swap) is also blue. Two blue neighbours remain. Blue is not freed for u.

Heawood (1890) constructs a specific map in which this occurs. The degree 4 argument was safe because distinct neighbour colours meant no two chains could share a colour. Degree 5 removes that guarantee. **The proof is wrong. The conjecture survives.**

Heawood's consolation prize: five colours

Five Colour Theorem (Heawood, 1890). *Every planar graph satisfies* $\chi(G) \leq 5$. – the theorem I first met in a UG graph theory class.

Proof.

Take a smallest planar G with $\chi(G) > 5$ and $\deg(u) \leq 5$. Colour the rest: $x_1, \ldots, x_5$ get $1, \ldots, 5$ (else done). If x_1, x_3 in *different* red–green chains: swap red↔green on x_1's chain, so x_1 becomes green, freeing red for u. Otherwise the red–green chain through x_1, u separates x_2 from x_4, so x_2, x_4 in *different* blue–yellow chains: swap, colour u blue. □

- The proof uses *five* colours. With 5 colours for 5 neighbours, all neighbour colours are distinct – so the red–green and blue–yellow chains involve four *different* colours and cannot share one. Heawood's interference cannot arise.
- This is a genuinely independent theorem, not a weakening of 4CT. It was proved in the same 1890 paper that showed Kempe's proof was wrong.

1912: Birkhoff and the chromatic polynomial

- George Birkhoff's paper *The reducibility of maps* (1913), directly addresses the Four Colour Conjecture. Rather than seeking a single colouring, he counts all of them.

The chromatic polynomial

$P(G, k)$ = number of proper colourings of G with k available colours. It is a polynomial in k.

- The Four Colour Conjecture becomes an analytic statement:

$$P(G, 4) > 0 \quad \text{for every planar } G.$$

- The approach via roots of P does not succeed, but the polynomial is central to everything that follows.

Deletion and contraction

Theorem

For any edge e of G,

$$P(G,k) \;=\; P(G\backslash e,\, k) \;-\; P(G/e,\, k).$$

Proof.

Colourings of $G\backslash e$ split into those giving the ends of e different colours (proper colourings of G) and those giving them the same colour (proper colourings of G/e). □

- Using deletion–contraction repeatedly: $P(K_3,k) = k(k-1)(k-2)$, and for a tree T on v vertices, $P(T,k) = k(k-1)^{v-1}$.
- Birkhoff and Lewis (1946): $P(G,x) > 0$ for all real $x \geq 5$, planar G. The conjecture that the same holds for all real $x > 4$ is open.
- Deletion–contraction is the recursion that Tutte, and matroid theory, will make universal.

1932: Whitney enters via colourings

- Whitney's early papers are *about map colouring*. In 1932, *A logical expansion in mathematics*:

$$P(G,k) \;=\; \sum_{A\subseteq E} (-1)^{|A|}\, k^{\,c(A)},$$

  a sum over edge subsets, $c(A)$ the number of components.
- Only the *rank function* $r(A) = v - c(A)$ of edge subsets matters – the polynomial depends on A only through r. Whitney observes this in 1932.
- 1935, *On the abstract properties of linear dependence* (1935): Whitney axiomatises the rank function directly, arriving at the notion of a *matroid* – which makes the subject of my PhD (2001), a by-product of map colouring.

Duality: where planarity really lives

- Every matroid M has a dual M^*: bases of M^* are the complements of bases of M.
- The *cycle matroid* $M(G)$ has edge sets of forests as its independent sets. A matroid isomorphic to some $M(G)$ is *graphic*; its dual $M(G)^*$ is *cographic*.

Theorem (Whitney's planarity criterion (1935))

G is planar if and only if $M(G)^*$ is graphic, in which case $M(G)^* = M(G^*)$ for a plane dual G^*.

- So the Four Colour problem lives *exactly* on the intersection of the graphic and cographic worlds.
- In a plane graph, interchanging faces and vertices gives a geometric duality of maps. Whitney showed this is the restriction to planar graphs of the abstract duality $M \mapsto M^*$: the two are the same thing.

1947: Tutte's dichromate

- Fix a linear order on E. Spanning tree T has *internally active* $e \in T$ if e is minimal in its fundamental cocycle; *externally active* $e \notin T$ if e is minimal in its fundamental cycle. Counts: $i(T)$, $e(T)$.

Tutte polynomial (activity definition, 1954)

$$T(G; x, y) = \sum_{T \text{ spanning tree}} x^{i(T)}\, y^{e(T)}.$$

- Whitney (1932) defined the *rank generating function*

$$R(G; x, y) = \sum_{A \subseteq E} x^{\,r(E)-r(A)}\, y^{\,|A|-r(A)}.$$

- Tutte showed $T(G; x, y) = R(G;\, x-1,\, y-1)$, so the two are equivalent. R (and hence T) depends only on the rank function: both are matroid invariants. $T(M^*; x, y) = T(M; y, x)$. See Oxley (2011) and Brylawski–Oxley (1992).
- Chromatic polynomial: $P(G, k) = (-1)^{r(E)} k^{c(G)}\, T(G;\, 1-k,\, 0)$. Setting $y = 0$ in the activity definition kills all terms with $e(T) > 0$, leaving only spanning trees with no externally active edges; the resulting sum evaluates to the chromatic polynomial via this identity.

Flows: the dual of colourings

Nowhere-zero k-flow

Orient the edges of G and assign each a value in $\mathbb{Z}_k \setminus \{0\}$ so that at every vertex the inflow equals the outflow.

Theorem (Tutte, 1954)

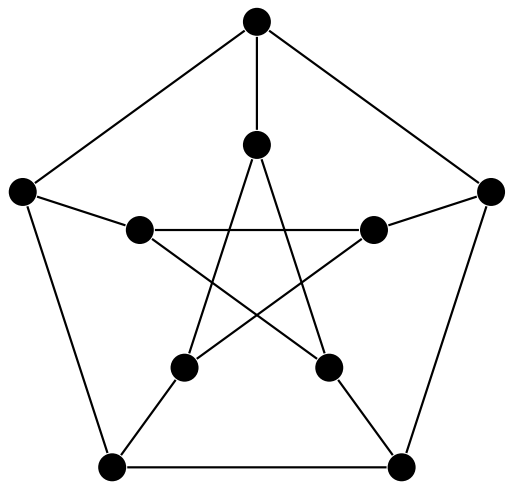
A plane graph G has a proper k-colouring if and only if its dual G^* has a nowhere-zero k-flow.

- Sketch: colour the faces of G^* with $\mathbb{Z}_k$; send each dual edge the *difference* of the colours across it. Proper colouring $\Leftrightarrow$ no difference is zero. Conservation at vertices is automatic.
- Flows are counted by $T(G; 0, 1-k)$: setting $x = 0$ kills all terms with $i(T) > 0$, leaving spanning trees with no internally active edges. The evaluation gives the flow polynomial, the mirror image of the chromatic polynomial under $x \leftrightarrow y$, consistent with matroid duality.

The Four Colour Theorem, dualised

The 4CT is equivalent to: every bridgeless planar graph has a nowhere-zero 4-flow. Flows make sense with *no planarity*. Tutte conjectured:

- **5-flow** Tutte (1954): every bridgeless graph has a nowhere-zero 5-flow.
- **4-flow** Tutte (1966): every bridgeless graph with no Petersen minor has a nowhere-zero 4-flow.
- **3-flow** Tutte (1972): every 4-edge-connected graph has a nowhere-zero 3-flow.



The Petersen graph: no nowhere-zero 4-flow.

All three remain open.

The *cycle double cover conjecture* (every bridgeless graph has cycles covering each edge twice) is implied by the 5-flow conjecture. Claimed proved by an AI in July 2026 (OpenAI, 2026) – see the coda.

The critical problem: colouring a matroid

- The Tutte polynomial generated an independent body of mathematics; we leave the historical thread briefly to follow it. For a graph, a proper k-colouring assigns colours to vertices so no edge is monochromatic. For a matroid there are no vertices – but there are circuits.
- **Colouring M over $\mathbb{F}_q$** (Crapo–Rota, 1970): a map $\phi : E \to \mathbb{F}_q^c$ such that no circuit of M maps to a linearly dependent set. For $M(G)$ with $c = 1$ this recovers proper vertex colouring: $\phi(e) = \phi(f)$ for an edge $e = \{u, v\}$ would make $\{u, v\}$ a circuit mapping to $\{0\}$.
- The *characteristic polynomial* $p(M; k) = (-1)^{r(E)} T(M; 1 - k, 0)$ satisfies $p(M(G); k) = P(G, k)$. The *critical exponent* $c(M)$: the least c with $p(M; q^c) > 0$, i.e. the minimum c for which a colouring over $\mathbb{F}_q^c$ exists.
- The Four Colour Theorem: $c(M(G)) \leq 2$ over $\mathbb{F}_2$ for all planar G, since $2^2 = 4$. Determining $c(M)$ for natural classes is wide open.

The Tutte polynomial meets coding theory

- Recall that $T(M_C)$ is a matroid invariant and depends only on the rank function of M_C. We can ask: which properties of C does it determine?
- A binary linear code $C \subseteq \mathbb{F}_2^n$ has a *vector matroid* M_C: columns of a generator matrix as ground set; independent sets are linearly independent columns. Note that $M_{C^\perp} = M_C^*$.

Theorem (Greene, 1976)

The weight enumerator $W_C(x,y) = \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)}$ *satisfies*

$$W_C(x,y) \;=\; (x-y)^{n-r}\, y^r \; T\left(M_C;\; \frac{x+y}{y},\; \frac{x}{x-y}\right), \qquad r = \dim C.$$

- So $T(M_C)$ determines W_C, the minimum distance, and $\dim C$.
- My PhD thesis explored which further properties of C are determined by $T(M_C; x, y)$, and which are not.

What $R(M_C; x, y)$ encodes

Let $N = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ over $\mathbb{F}_2$. Coefficients of $R(M_N; x, y)$:

	x^3	x^2	x^1	x^0
y^0	1	4	6	3
y^1			1	1

$R(M; 0, 0) = 3$ bases (entry x^0y^0: sets with $r(A) = |A| = r(E)$).
$R(M; 1, 0) = 1 + 4 + 6 + 3 = 14$ independent sets (row y^0 only; $y = 0$ kills terms with $|A| > r(A)$).

Theorem (Maximum strength, Rutherford 2001)

Index the anti-diagonals of the coefficient array by $s =$ (power of x) $-$ (power of y). On anti-diagonal s, the entries are $a_{s,0}$, $a_{s+1,1}$, $a_{s+2,2}, \ldots$ The maximum strength of C is $l = k - r$, where r is the least s such that $a_{s,0} \neq 0$ and $a_{s+t,t} = 0$ for all $t \geq 1$.

Anti-diagonal $s = 0$: $a_{0,0} = 3$ and $a_{1,1} = 1 \neq 0$ – condition fails.
Anti-diagonal $s = 1$: $a_{1,0} = 6 \neq 0$ and $a_{2,1} = a_{3,2} = \cdots = 0$ – condition holds. So $r = 1$, $l = k - 1 = 2$: every pair of coordinates takes each 2-tuple equally often.

MacWilliams from matroid duality

- We have $M_{C^\perp} = M_C^*$ and $T(M^*; x, y) = T(M; y, x)$.
- Applying Greene's theorem to both C and $C^\perp$ and using duality gives the weight enumerator of $C^\perp$ in terms of that of C:

Theorem (MacWilliams, 1963)

$$W_{C^\perp}(x, y) \;=\; \frac{1}{|C|}\, W_C(x + y,\, x - y).$$

- This classical result – usually proved by character sums – follows directly from $T(M^*; x, y) = T(M; y, x)$ via Greene.
- Recall Tutte (1954): colourings of a plane graph correspond to flows in its dual. The MacWilliams identity is the same duality, applied to codes via the Tutte polynomial.

The Rutherford polynomial

- $\mathbb{Z}_4$-linear codes do not fit the vector matroid framework directly: they need a *pair* of binary matroids (M_1, M_2), one for each level of the chain $C_1 \subseteq C_2$.
- My PhD thesis (2001) defines the *Rutherford polynomial* of such a pair:

$$G(M_1, M_2;\, v, x, y) = \sum_{A \subseteq B \subseteq E} v^{|B|-|A|}\, x^{r_1(E)-r_1(B)}\, y^{|A|-r_2(A)},$$

  a three-variable generalisation of the rank generating function.
- P.J. Cameron (2002) describes this as giving "some features of C as specialisations" – including the weight enumerators of both C_1 and C_2 – though not the full symmetrised weight enumerator of the $\mathbb{Z}_4$-code.
- This shows the limits of the matroid approach: a single matroid is not enough to capture $\mathbb{Z}_4$-linear codes.

What the Tutte polynomial does not know

Covering radius

The *covering radius* $r(C)$ of a code $C \subseteq \mathbb{F}_2^n$ is $\max_{v \in \mathbb{F}_2^n} d(v, C)$: the worst-case error-correction shortfall.

- At BCC17 (Canterbury, 1999) I posed, with Fuad Shareef: is $r(C)$ determined by $T(M_C; x, y)$? Published as Problem 361 in *Discrete Math.* 231 (2001), with P.J. Cameron as correspondent (2001).

Theorem (Britz and Rutherford, 2005)

The covering radius of a binary linear code is not in general determined by the Tutte polynomial of the associated matroid.

- Two $[11, 6]$ binary codes share a Tutte polynomial but have covering radii 2 and 3. The example comes from a pair of graphs with equal Tutte polynomials noted by Tutte (1974), applied to their bond codes.
- The polynomial lives at the level of the matroid; the matroid forgets the metric geometry of the code.

Heesch: industrialising Kempe

We return to the historical thread: the conjecture is still open in 1969.

- The plan never changed since 1879: find an *unavoidable set* of configurations, each *reducible*.
- **Unavoidable**: every smallest counterexample must contain one of them. (Kempe's set: a vertex of degree 3, 4 or 5.)
- **Reducible**: any smallest counterexample containing it can be shrunk, contradiction. (Kempe managed degrees 3 and 4.)
- Heinrich Heesch (1969) develops *discharging* to generate unavoidable sets systematically, and estimates the problem needs about 10,000 configurations. Too many for a human. Perhaps not for a machine.

Discharging in miniature

Lemma

Every planar triangulation with minimum degree 5 *contains two adjacent vertices of degree* 5*, or a degree* 5 *vertex adjacent to a degree* 6 *vertex.*

Proof sketch.

Give each vertex charge $6 - \deg(v)$. Total charge $= 12 > 0$ (since $e = 3v - 6$ in a triangulation). *Discharging rule*: each degree 5 vertex sends $\frac{1}{5}$ to each neighbour of degree ≥ 7. If the lemma failed, every vertex ends with charge ≤ 0: contradiction. □

- Euler guarantees positive total charge; the discharging rules force it to concentrate; the configuration it concentrates on is unavoidable.
- Reducibility of each configuration is proved using Kempe chain arguments – Kempe's tool survives. The actual proof uses hundreds of discharging rules.

June 1976: Appel and Haken

- Kenneth Appel and Wolfgang Haken, with John Koch, at the University of Illinois Appel–Haken (1977).
- An unavoidable set of 1,936 reducible configurations, found by hand-tuned discharging; reducibility checked by computer, around 1,200 hours of machine time.
- Announced June 1976. The Illinois maths department franks its post: four colors suffice.
- By the autumn term, when my geography teacher stated the theorem as fact, the proof was a few months old.

Is it a proof?

- No human has read the proof. No human *can* read the proof.
- Tymoczko (1979): a proof must be *surveyable* – a mathematician must be able to check it step by step. The computer verification is not surveyable; it is an empirical observation, not a logical argument.
- Reply: the hand-checked parts of Appel–Haken contained *more* errors (all repairable) than the machine-checked parts. Human surveyability is no guarantee of correctness.
- The controversy was the first serious philosophical debate about computer-assisted proof, and it has never entirely gone away. Fifty years later it takes a new form: see the coda.

Cleaning up, then certainty

- 1997: Robertson, Sanders, Seymour and Thomas re-prove the theorem with 633 configurations and fully automated discharging (1997).
- 2005: Georges Gonthier formalises the entire proof in the proof assistant Coq; published as Gonthier (2008). The only trusted component is Coq's small kernel.
- The theorem once doubted for relying on computers is now among the most rigorously verified results in mathematics, precisely because of the Coq formalisation.

The living descendant: Hadwiger

Conjecture (Hadwiger, 1943)

Every graph with no K_t minor is $(t-1)$-colourable.

- $t = 5$ is equivalent to the Four Colour Theorem Wagner (1937).
- $t = 6$: Robertson, Seymour and Thomas (1993), by reduction to the Four Colour Theorem.
- $t \geq 7$: open. Widely considered one of the deepest open problems in graph theory.

On the matroid side, Tutte's flow conjectures and the critical problem remain wide open – though the cycle double cover conjecture, implied by the 5-flow conjecture, was claimed in July 2026 by an AI system. Guthrie's question about counties is settled; the problems it generated are not.

Coda: July 2026

The *cycle double cover conjecture* – every bridgeless graph has cycles covering each edge exactly twice – was considered one of the deepest open problems in graph theory. It is implied by Tutte's 5-flow conjecture and holds for planar graphs by the 4CT.

- On 9 July 2026, OpenAI published a claimed proof (OpenAI, 2026), attributed to GPT-5.6 Sol Ultra running 64 concurrent subagents, with a Lean formalisation alongside.
- An exposition by Oum (2026) treats the proof as correct. At the time of this talk it has not appeared in a refereed journal.

Appel and Haken used a computer to check cases that humans had identified. Here the proof strategy, key lemmas, and verification are all machine-generated. Tymoczko's surveyability criterion (1979) was already strained by Appel–Haken; the CDC proof breaks it entirely. Whether it counts as a proof is, again, a question about what we mean by that word.

Back to the classroom

- Autumn 1976: a geography teacher states the theorem as fact, to a class that includes one future matroid theorist. The proof had been announced in June; perhaps the teacher had read about it.
- Fifty years on, the Four Colour Theorem is the most famous theorem whose proof no one has read.
- The mathematics it generated – chromatic polynomials, matroids, the Tutte polynomial, nowhere-zero flows, discharging, formal verification – has proved more consequential than the theorem itself. For an accessible account of the whole story, see Wilson (2002).

Thank you. Questions?

References

- Kempe, *Amer. J. Math.* 2 (1879), 193–200.
- Heawood, *Quart. J. Pure Appl. Math.* 24 (1890), 332–338.
- Birkhoff, *Amer. J. Math.* 35 (1913), 115–128.
- Whitney, A logical expansion, *Bull. AMS* 38 (1932), 572–579.
- Whitney, *Amer. J. Math.* 57 (1935), 509–533.
- Wagner, *Math. Ann.* 114 (1937), 570–590.
- Hadwiger, *Vjschr. Naturf. Ges. Zürich* 88 (1943), 133–142.
- Tutte, *Canad. J. Math.* 6 (1954), 80–91.
- MacWilliams, *Bell Syst. Tech. J.* 42 (1963), 79–94.
- Tutte, *J. Combin. Theory* 1 (1966), 15–50.
- Heesch, *Untersuchungen zum Vierfarbenproblem*, 1969.
- Crapo–Rota, *Combinatorial Geometries*, MIT Press, 1970.
- Tutte, *Aequationes Math.* 9 (1972), 1–11.
- Greene, *Studia Appl. Math.* 55 (1976), 119–128.
- Appel–Haken, *Illinois J. Math.* 21 (1977), 429–567.
- Tymoczko, *J. Philosophy* 76 (1979), 57–83.
- Brylawski–Oxley, in *Matroid Applications*, CUP, 1992.
- Robertson–Seymour–Thomas, *Combinatorica* 13 (1993), 279–361.
- Robertson–Sanders–Seymour–Thomas, *JCTB* 70 (1997), 2–44.
- Rutherford, PhD thesis, Queen Mary, 2001.
- Cameron, *Discrete Math.* 231 (2001), 469–478.
- Cameron, lecture notes, Queen Mary, 2002.
- Wilson, *Four Colours Suffice*, Penguin, 2002.
- Britz–Rutherford, *Discrete Math.* 296 (2005), 117–120.
- Gonthier, *Notices AMS* 55 (2008), 1382–1393.
- Oxley, *Matroid Theory*, OUP, 2011.
- Whitty, `www.theoremoftheday.org`.
- OpenAI, cycle double cover conjecture, preprint, 2026.
- Oum, arXiv:2607.16356, 2026.