

On Representatives of Subsets

P. Hall, *Journal of the London Mathematical Society*, 1935

A Landmark in Combinatorics

P. Hall (1935)

Presented by Carrie Rutherford

Maths Study Group, 13th March 2026

Overview

- 1 Historical Context
- 2 The Main Theorem
- 3 The Proof
- 4 Worked Examples
- 5 Theorem 2 and Theorem 3
- 6 Modern Reformulations and Applications
- 7 Generalisations
- 8 Discussion and Legacy

Philip Hall (1904–1982)

- Fellow of King's College, Cambridge
- One of the leading algebraists of the twentieth century; major contributions to group theory (Hall subgroups, Hall–Higman theorem, p -groups)
- This 1935 paper is a rare excursion into *combinatorics* – and produced one of the most widely used theorems in discrete mathematics
- Paper submitted 23 April 1934; read before the LMS 26 April 1934

The paper in brief

- 4.5 pages
- 3 theorems + 1 lemma
- Elegant inductive proof
- Generalises König's theorem

Antecedents: König's Theorem

Setting

A set S of mn elements is partitioned into m classes of n elements each in *two* independent ways, say the (a) -classes and the (b) -classes.

Theorem (König, *Berliner Sitzungsberichte* 32 (1933))

*There always exists a set R of m elements of S that is simultaneously a **complete system of representatives** (CSR) for the (a) -classes and a CSR for the (b) -classes.*

In König's setting the classes are disjoint, so any system of representatives is automatically a set of distinct elements – CSR and CDR coincide. Hall's problem drops the disjointness assumption, so a CDR (distinct representatives) becomes a genuine additional requirement.

- Originally proved as a theorem about bipartite graphs
- Hall's paper *subsumes* this as a corollary to a more general criterion

König's Theorem — Example

Let $S = \{1, 2, 3, 4, 5, 6\}$ with $m = 3$, $n = 2$. Two partitions into classes of size 2:

(a)-classes

$$A_1 = \{1, 2\}$$

$$A_2 = \{3, 4\}$$

$$A_3 = \{5, 6\}$$

(b)-classes

$$B_1 = \{1, 3\}$$

$$B_2 = \{2, 5\}$$

$$B_3 = \{4, 6\}$$

A simultaneous CSR: $R = \{2, 3, 6\}$

$$2 \in A_1, \quad 3 \in A_2, \quad 6 \in A_3 \quad (a)\text{-CSR } \checkmark$$

$$2 \in B_2, \quad 3 \in B_1, \quad 6 \in B_3 \quad (b)\text{-CSR } \checkmark$$

The non-trivial content: not every choice works. For instance $R' = \{1, 3, 5\}$ is an (a)-CSR but $1, 3 \in B_1$ so it fails as a (b)-CSR.

Hall's Problem

Given a finite family of subsets T_1, \dots, T_m of a set S (arbitrary sizes, arbitrary overlaps), when does there exist a **complete system of distinct representatives** (CDR)?

Definition

A **CDR** for (T_1, \dots, T_m) is m *distinct* elements $a_1, \dots, a_m \in S$ with $a_i \in T_i$ for each i .

Key generalisations over König:

- The sets T_i need not have equal cardinality
- The sets may overlap arbitrarily
- S itself need not be finite

The Marriage Condition

Necessary condition (obvious)

If a CDR exists, then for every $J \subseteq \{1, \dots, m\}$,

$$\left| \bigcup_{i \in J} T_i \right| \geq |J|.$$

One cannot find $|J|$ distinct representatives from fewer than $|J|$ elements.

Hall's insight: **this obvious necessary condition is also sufficient.**

Hall's Marriage Theorem

Theorem (Hall, 1935 – Theorem 1)

In order that a CDR of (T_1, \dots, T_m) shall exist, it is sufficient that for each $k = 1, \dots, m$, every selection of k sets contains between them at least k elements of S .

That is, the **Hall condition**

$$\left| \bigcup_{i \in J} T_i \right| \geq |J| \quad \text{for every } J \subseteq \{1, \dots, m\}$$

is *necessary and sufficient* for a CDR to exist.

Simple to state. Elegant to prove. Universally applicable.

Why “Marriage Theorem”?

The marriage interpretation

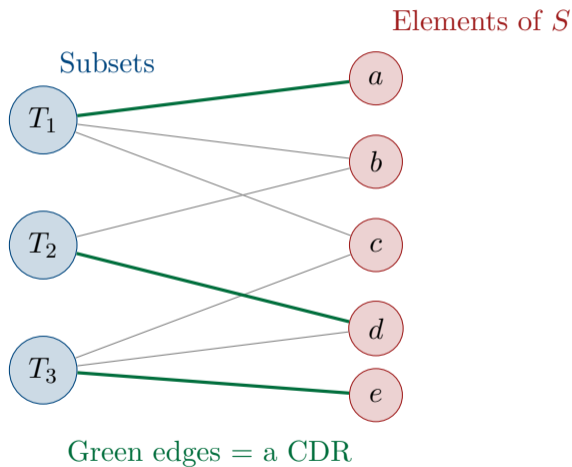
Think of each T_i as the set of acceptable partners for person i .

A **CDR** is a matching that saturates the people: each person i gets a distinct acceptable partner, though some potential partners in S may go unchosen.

The Hall condition: every group of k people collectively has at least k acceptable partners – otherwise some subset is “unmarriageable.”

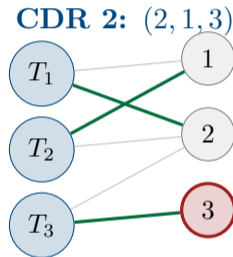
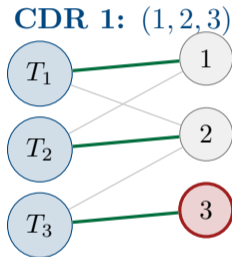
The name is due to the combinatorial folklore that grew around Hall’s result in the late 1930s and 1940s; Hall himself did not use it.

The Marriage Theorem — Bipartite Graph Picture



The Kernel — Example

$S = \{1, 2, 3\}$, $T_1 = \{1, 2\}$, $T_2 = \{1, 2\}$, $T_3 = \{2, 3\}$. Two CDRs:



Red node = element 3: matched in *every* CDR $\Rightarrow 3 \in R$.

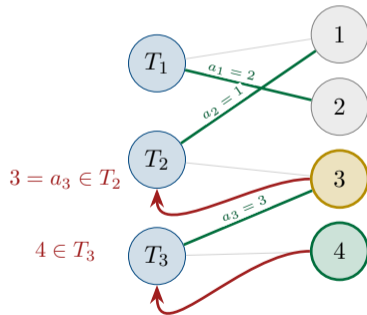
Elements 1 and 2 each miss one CDR $\Rightarrow 1, 2 \notin R$.

Kernel $R = \{3\}$

Switching Chain – Example

Given a CDR (a_1, \dots, a_m) , a **switching chain** starting at $a \notin \{a_1, \dots, a_m\}$ is a sequence $a \in T_i, a_i \in T_j, \dots$. The chain may be stopped at any point to yield a new CDR.

Example: $T_1 = \{1, 2\}, T_2 = \{1, 3\}, T_3 = \{3, 4\}$. CDR $(2, 1, 3)$. **Introduce** $4 \in T_3$.



Chain:

$4 \in T_3 \rightarrow 3 = a_3 \in T_2$
 $\rightarrow 1 = a_2 \in T_1$

Stop at T_3 :

$(2, 1, 4) \checkmark$

Stop at T_2 :

$(2, 3, 4) \checkmark$

Stop at T_1 :

$(1, 3, 4) \checkmark$

The Key Lemma

Lemma (Hall, 1935)

Let $R = \{a_1, \dots, a_p\}$ be the kernel of (T_1, \dots, T_m) , indexed so that a_i is the representative of T_i in every CDR. Then T_1, \dots, T_p contain between them *exactly* p elements, namely a_1, \dots, a_p .

(Such an indexing always exists: it is a relabelling of whichever sets have a forced representative.)

Proof idea. Fix any CDR and let R' be the set of representatives that cannot be displaced – meaning no switching chain can replace them while preserving the CDR property.

Show $R' = R$ via two steps:

- 1 $R \subseteq R'$: if $a_j \in R$ could be displaced, a switching chain would yield a CDR omitting a_j , contradicting $a_j \in R$.
- 2 $R' \subseteq R$: if a_j cannot be displaced, T_1, \dots, T_p are collectively tight – every element is already spoken for, so nothing can be swapped in from outside, forcing a_j into every CDR.

Key Lemma — Example

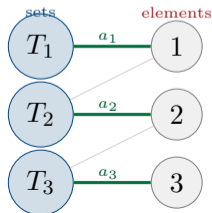
Example. $T_1 = \{1\}$, $T_2 = \{1, 2\}$,
 $T_3 = \{2, 3\}$.

Unique CDR: $(1, 2, 3)$, so

$R = \{1, 2, 3\}$, $p = 3$.

$T_1 \cup T_2 \cup T_3 = \{1, 2, 3\}$ — exactly 3
elements. ✓

Every element is pinned: no slack
anywhere.



Hall (1935)

The lemma's force is in the *tight* case: p sets collectively covering exactly p elements, leaving no slack. This example achieves the tightest possible bound at every step.

Why this matters. In the proof of Thm 1, if $T_m \subseteq R^*$ (the kernel of the first $m - 1$ sets), then by the Lemma those p tight sets together with T_m give $p + 1$ sets covering only p elements — contradicting Hall's condition. So $T_m \not\subseteq R^*$ and the CDR can always be extended by one more representative.

Contrast. Our running example $T_1 = \{1, 2\}$, $T_2 = \{1, 3\}$, $T_3 = \{3, 4\}$ has kernel $R = \{3\}$, but no sub-family achieves the tight bound — Hall's

Proof of Theorem 1 (Induction on m)

Inductive hypothesis

Assume the theorem holds for any family of $m - 1$ subsets.

Base case ($m = 1$): Trivial — Hall's condition forces $T_1 \neq \emptyset$.

Inductive step. Assume the Hall condition holds for (T_1, \dots, T_m) .

- 1 By induction, (T_1, \dots, T_{m-1}) has at least one CDR.
- 2 It suffices to show $T_m \not\subseteq R^*$, where R^* is the kernel of all CDRs of (T_1, \dots, T_{m-1}) .
- 3 If $T_m \subseteq R^*$: the Lemma gives $|T_1 \cup \dots \cup T_p| = p$; since $T_m \subseteq R^* = T_1 \cup \dots \cup T_p$, adding T_m contributes no new elements. So $p + 1$ sets cover only p elements — contradicting Hall's condition.
- 4 Hence $\exists a_m \in T_m \setminus R^*$. Take a CDR of (T_1, \dots, T_{m-1}) avoiding a_m ; append a_m .
 \square

Example 1: Condition Fails

Let $S = \{1, 2, 3\}$ and consider

$$T_1 = \{1, 2\}, \quad T_2 = \{1, 2\}, \quad T_3 = \{1, 2\}.$$

Check the Hall condition: Take $J = \{1, 2, 3\}$:

$$T_1 \cup T_2 \cup T_3 = \{1, 2\}, \quad |\cdot| = 2 < 3 = |J|.$$

Conclusion

Hall's condition fails, so no CDR exists. Indeed, we cannot pick three distinct elements all from $\{1, 2\}$.

Example 2: Condition Holds

Let $S = \{1, 2, 3, 4\}$ and $T_1 = \{1, 2\}$, $T_2 = \{2, 3\}$, $T_3 = \{1, 3, 4\}$.

Verify the Hall condition (singletons trivially satisfied; non-trivial cases):

Sub-collection	Union	Holds?
$\{T_1\}, \{T_2\}, \{T_3\}$	size ≥ 2 each	✓
$\{T_1, T_2\}$	$\{1, 2, 3\}$, size 3	$3 \geq 2$ ✓
$\{T_1, T_3\}$	$\{1, 2, 3, 4\}$, size 4	$4 \geq 2$ ✓
$\{T_2, T_3\}$	$\{1, 2, 3, 4\}$, size 4	$4 \geq 2$ ✓
$\{T_1, T_2, T_3\}$	$\{1, 2, 3, 4\}$, size 4	$4 \geq 3$ ✓

A valid CDR

$$a_1 = 2, \quad a_2 = 3, \quad a_3 = 1.$$

Example 3: Latin Squares / Timetabling

Application

A school must schedule m teachers, each willing to teach certain subjects (their T_i). Can each teacher be assigned a *distinct* subject?

Hall's theorem gives an exact criterion: it is possible if and only if every group of k teachers collectively covers at least k distinct subjects.

Equivalently: A *system of distinct representatives* exists for the rows of a 0–1 matrix iff the Hall condition holds for the columns covered by each row.

This is the starting point for the theory of **Latin transversals** and **Latin squares**.

Theorem 2: Distinct Representatives from Disjoint Classes

Theorem (Hall, 1935 – Theorem 2)

Let $S = S_1 \cup S_2 \cup \dots$ be a partition of S into disjoint classes. Then there exist m elements a_1, \dots, a_m with no two in the same class and $a_i \in T_i$, if and only if for each k , any k of the sets T_i together contain elements from at least k distinct classes.

Proof idea

Replace each T_i by $t_i = \{S_j : S_j \cap T_i \neq \emptyset\}$, the collection of classes meeting T_i . The condition translates into Hall's condition for the t_i , and Theorem 1 gives m distinct classes S_{j_1}, \dots, S_{j_m} ; pick one a_i from each $S_{j_i} \cap T_i$.

Theorem 3: Common Representatives for Two Classifications

Theorem (Hall, 1935 – Theorem 3)

Let S be partitioned in two ways, $S = S_1 \vee \cdots \vee S_m = S'_1 \vee \cdots \vee S'_m$ (disjoint unions). If for every k , any k of the S'_j -classes together contain elements from at least k of the S_i -classes, then there exist m elements a_1, \dots, a_m (after possibly reindexing the S'_j) with $a_i \in S_i \cap S'_i$ for each i .

Corollary: König's Theorem

When all classes have the same size n , the condition holds automatically by a counting argument, recovering König's theorem (1933).

Graph-Theoretic Reformulation

Definition

A **bipartite graph** $G = (X \cup Y, E)$ has a **perfect matching** (from X into Y) if there exist $|X|$ pairwise disjoint edges covering X .

Theorem (Hall's Theorem — Graph Version)

A bipartite graph $G = (X \cup Y, E)$ has a perfect matching from X into Y if and only if

$$|N(A)| \geq |A| \quad \text{for every } A \subseteq X,$$

where $N(A) = \{y \in Y : xy \in E \text{ for some } x \in A\}$ is the neighbourhood of A .

Dictionary: $X = \{T_1, \dots, T_m\}$, $Y = S$, edges encode membership, CDR \leftrightarrow matching saturating X (not necessarily Y).

Deficiency Version (Ore, 1955)

When Hall's condition fails, how far short can we fall?

Definition

The **deficiency** of the family is

$$\text{def}(T_1, \dots, T_m) = \max_{J \subseteq [m]} \left(|J| - \left| \bigcup_{i \in J} T_i \right| \right).$$

Theorem (Deficiency Version)

The maximum size of a system of partial distinct representatives is $m - \text{def}(T_1, \dots, T_m)$.

This quantifies precisely how far the family is from having a CDR.

Pure mathematics

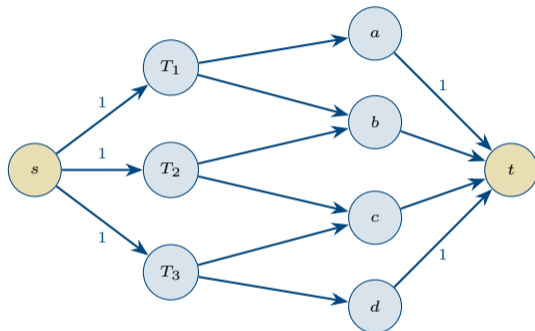
- Latin squares and transversals
- Doubly stochastic matrices
(Birkhoff–von Neumann theorem)
- k -regular bipartite graphs are
 k -edge-colourable
- Factor theory in group algebras
- Dilworth's theorem on partially
ordered sets

Applied settings

- Scheduling and timetabling
- Assignment problems (HR, tasks)
- Network flow (max-flow min-cut)
- Coding theory (MDS codes)
- Cryptography (key distribution)
- Constraint satisfaction

“Hall’s theorem is to combinatorics what the intermediate value theorem is to analysis: simple to state, deep to prove, and everywhere in use.”

Connection to Max-Flow Min-Cut



Hall's condition \Leftrightarrow max-flow = $m \Leftrightarrow$ min-cut = m (König–Egerváry).

Algorithmic Aspects

Hall's theorem is *existential*; constructive algorithms came later:

Algorithm	Reference	Complexity
Augmenting paths	Berge (1957)	$O(mn)$
Hungarian method	Kuhn (1955)	$O(m^3)$
Hopcroft–Karp	Hopcroft & Karp (1973)	$O(m^{1/2}n)$
Randomised matching	Schwartz–Zippel	$O(mn^\omega)$

The **augmenting path** method is precisely the *switching chain* argument from Hall's Lemma, made algorithmic.

Rado's Theorem (1942)

Characterises when a linear system has a solution rainbow across a partition. Subsumes Hall's theorem.

Infinite Hall (M. Hall Jr., 1948)

Hall's condition suffices for infinite families when each T_i is finite; full generality needs the axiom of choice.

b -Matchings & Matroids

Analogues for b -matchings (each element used $\leq b$ times) and matroid intersection (Edmonds).

Haxell's Theorem (1995)

When $|T_i| \geq 2\Delta$ ($\Delta = \max$ overlap degree), guarantees a rainbow matching in hypergraphs.

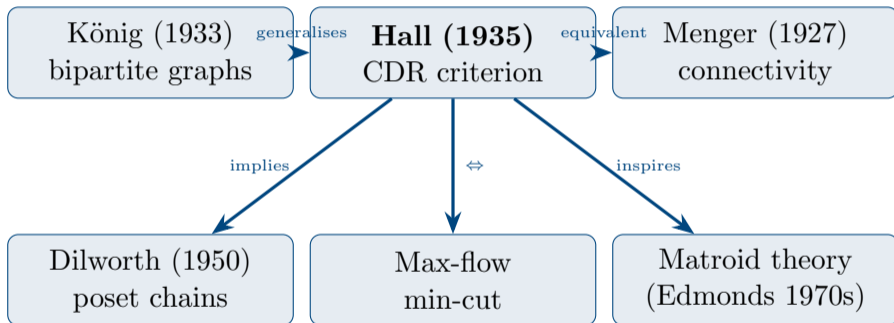
What Makes the Proof Beautiful?

- ① **Purity of induction.** The proof proceeds by induction on m with a single, clean case split.
- ② **The Lemma does the work.** Characterising the kernel R via switching chains is the genuinely clever observation; the theorem then follows almost immediately.
- ③ **No auxiliary structure.** Unlike König's original proof, no graph theory or matrix algebra is needed – only the definitions.
- ④ **Tight proof of necessity and sufficiency.** Both directions are elementary once the framework is set up.
- ⑤ **Four pages.** The entire argument, including Theorems 2 and 3, fits in four and a half journal pages. A masterclass in concision.

Legacy and Impact

- Hall's Marriage Theorem appears in virtually every combinatorics textbook
- Cited in graph theory, algebra, analysis, probability, and theoretical computer science
- Equivalent to König's minimax theorem, Dilworth's chain decomposition theorem, and the max-flow min-cut theorem (all via Menger's theorem) – a remarkable cluster of equivalences
- The concept of a *system of distinct representatives* (SDR) became a fundamental object of study; see the monograph by Mirsky (1971)
- Motivated the entire theory of *transversals in set systems* and, later, *matroid theory* (Edmonds, Welsh)
- One of the earliest examples of an “if and only if” criterion in combinatorics with a structural proof

Summary



For discussion

- 1 Verify directly that Hall's condition is equivalent to the König–Egerváry theorem on the size of a maximum matching in a bipartite graph.
- 2 Prove that a k -regular bipartite graph ($k \geq 1$) satisfies Hall's condition, hence has a perfect matching.
- 3 State and prove the deficiency formula.
- 4 Hall's condition is necessary and sufficient. Is there a polynomial-time algorithm to *find* a CDR when one exists? Describe it.
- 5 The Lemma characterises the kernel R . What is the structure of the *set of all CDRs* of a family?

References

- P. Hall, “On representatives of subsets,” *J. London Math. Soc.* **10** (1935), 26–30.
- D. König, “Graphok és matrixok,” *Berliner Sitzungsberichte* **32** (1933) [cited by Hall].
- B.L. van der Waerden, “Ein Satz über Klasseneinteilungen von endlichen Mengen,” *Abh. Math. Sem. Hamburg* **5** (1927), 185.
- M. Hall Jr., “Distinct representatives of subsets,” *Bull. Amer. Math. Soc.* **54** (1948), 922–926.
- L. Mirsky, *Transversal Theory*, Academic Press, 1971.
- J. Hopcroft & R.M. Karp, “An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs,” *SIAM J. Comput.* **2** (1973), 225–231.
- R. Rado, “A theorem on independence relations,” *Quart. J. Math. Oxford* **13** (1942), 83–89.
- P. Erdős & A. Rényi, “On random graphs,” *Publ. Math. Debrecen* **6** (1959).
- T. Haxell, “A condition for matchability in hypergraphs,” *Graphs Combin.* **11** (1995).
- J. Diestel, *Graph Theory*, 5th ed., Springer, 2017.

Thank you

Hall's Marriage Theorem, 1935

“In order that a CDR of (T_1, \dots, T_m) shall exist, it is sufficient that, for each $k = 1, \dots, m$, any selection of k of the sets shall contain between them at least k elements of S .”

– P. Hall, 1935