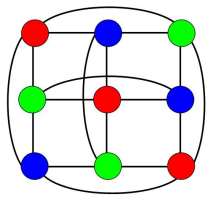
Flows, Not Colours

Nowhere-Zero Flows and the Cycle Double Cover Conjecture

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LSBU Maths Study Group

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Where we left off in July

- Recall the Four Colour Theorem talk: a planar graph is k-face-colourable if and only if it admits a nowhere-zero k-flow.
- Today we look at the *flow* side of that equivalence directly, on its own terms.
- A key example along the way: the **Petersen graph**, cubic and 3-edge-connected, the graph every flow theorist eventually has to think about.
- And we will finish by returning to something flagged as breaking news in July: the 2026 claimed proof of the **Cycle Double Cover Conjecture**, which turns out to use exactly the machinery built today.

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This is exactly Kirchhoff's current law: read $f(e)$ as the current on edge e, and this says current into a junction equals current out. An assignment satisfying conservation at every vertex is called a **flow** on G.

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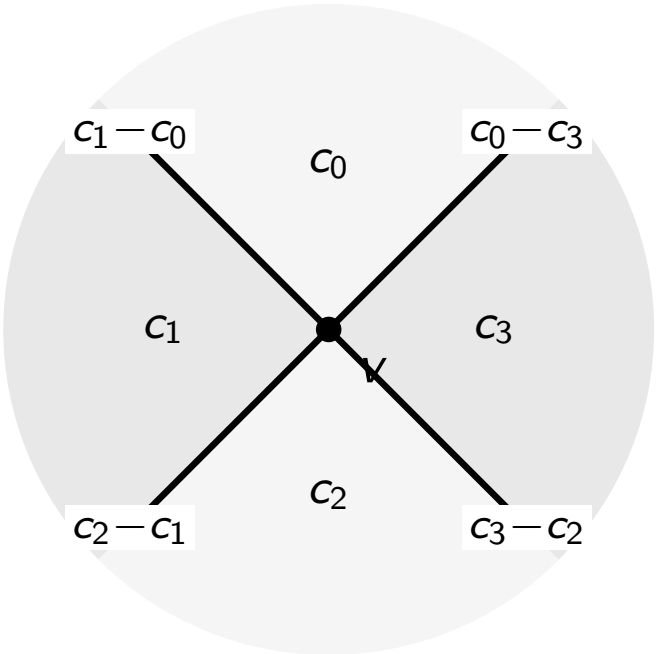
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A nowhere-zero k-flow is such an assignment with values from $\{1, \ldots, k-1\}$ (or a group of order k, minus the identity).

From colourings to flows

Start from a proper k-face-colouring of G (adjacent faces get different colours), and build a flow directly on G.



- Walk round v: four edges, four faces between them, coloured c_0, c_1, c_2, c_3 in order.
- Each edge separates two consecutive faces; give it (next colour) $-$ (previous colour).
- Properness ($c_i \neq c_{i+1}$) is exactly nowhere-zero.
- **A flow**: round v, $(c_1 - c_0) + (c_2 - c_1) + (c_3 - c_2) + (c_0 - c_3) = 0$. Same at every vertex.

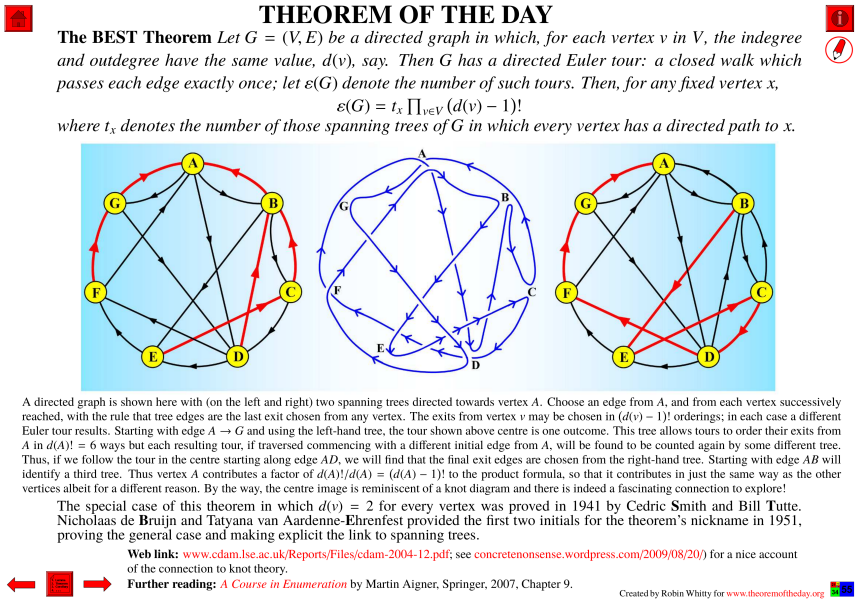
A fact worth flagging now

Non-obvious fact

Whether G has a nowhere-zero k-flow depends only on the *order* of the group Γ, not which group of that order you pick.

- This is a theorem of Tutte: G has a nowhere-zero Γ-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow, where $k = |\Gamma|$.
- We are not proving this today, but the reason will become visible later: the *number* of nowhere-zero Γ-flows turns out to be a polynomial in $|\Gamma|$ alone (the flow polynomial, coming up).
- A second, easier independence: the *orientation* does not matter either. Reversing an edge and negating its value keeps conservation and keeps it nowhere-zero. So only $|\Gamma|$ is doing any work, not Γ itself and not the orientation.

Aside: the BEST theorem (Theorem of the Day)



THEOREM OF THE DAY

The BEST Theorem *Let $G = (V, E)$ be a directed graph in which, for each vertex v in V, the indegree and outdegree have the same value, $d(v)$, say. Then G has a directed Euler tour: a closed walk which passes each edge exactly once; let $\varepsilon(G)$ denote the number of such tours. Then, for any fixed vertex x,*

$$\varepsilon(G) = t_x \prod_{v \in V} (d(v) - 1)!$$

where t_x denotes the number of those spanning trees of G in which every vertex has a directed path to x.

A directed graph is shown here with (on the left and right) two spanning trees directed towards vertex A. Choose an edge from A, and from each vertex successively reached, with the rule that tree edges are the last exit chosen from any vertex. The exits from vertex v may be chosen in $(d(v) - 1)!$ orderings; in each case a different Euler tour results. Starting with edge $A \to G$ and using the left-hand tree, the tour shown above centre is one outcome. This tree allows tours to order their exits from A in $d(A)! = 6$ ways but each resulting tour, if traversed commencing with a different initial edge from A, will be found to be counted again by some different tree. Thus, if we follow the tour in the centre starting along edge AD, we will find that the final exit edges are chosen from the right-hand tree. Starting with edge AB will identify a third tree. Thus vertex A contributes a factor of $d(A)!/d(A) = (d(A) - 1)!$ to the product formula, so that it contributes in just the same way as the other vertices albeit for a different reason. By the way, the centre image is reminiscent of a knot diagram and there is indeed a fascinating connection to explore!

The special case of this theorem in which $d(v) = 2$ for every vertex was proved in 1941 by Cedric **S**mith and Bill **T**utte. Nicholaas de **B**ruijn and Tatyana van Aardenne-**E**hrenfest provided the first two initials for the theorem's nickname in 1951, proving the general case and making explicit the link to spanning trees.

Web link: www.cdam.lse.ac.uk/Reports/Files/cdam-2004-12.pdf; see concretenonsense.wordpress.com/2009/08/20/) for a nice account of the connection to knot theory.

Further reading: *A Course in Enumeration* by Martin Aigner, Springer, 2007, Chapter 9.

Created by Robin Whitty for www.theoremoftheday.org

- Tutte turns up again elsewhere: the **BEST theorem** (de Bruijn, van Aardenne-Ehrenfest, Smith, Tutte, 1941/1951) counts directed Euler tours in a *balanced* digraph, one where every vertex has in-degree equal to out-degree.
- That balance condition is exactly our conservation law, for the simplest possible flow: the all-ones flow in $\mathbb{Z}$. A balanced orientation *is* a nowhere-zero flow.
- Full statement and history: `theoremoftheday.org`, "The BEST Theorem".

Does every graph have one?

- **Bridges kill it.** If e is a bridge, removing it splits G into two pieces; all flow crossing between them must pass through e, forcing $f(e) = -f(e)$, so $f(e) = 0$.

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- **Bridges kill it.** If e is a bridge, removing it splits G into two pieces; all flow crossing between them must pass through e, forcing $f(e) = -f(e)$, so $f(e) = 0$.
- **Necessary and sufficient:** G has a nowhere-zero k-flow for *some* k if and only if G is bridgeless.
- So existence of *some* nowhere-zero flow is a solved, simple question. The hard, partly open question is the *minimum* k.

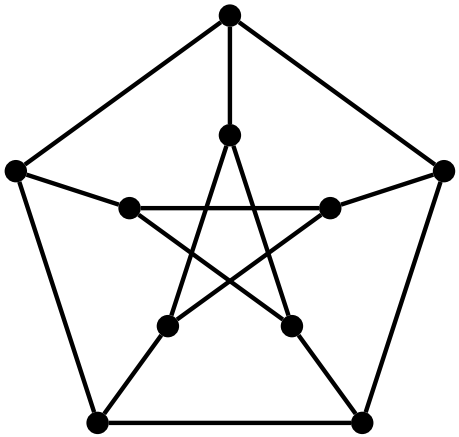
Why bridgeless is sufficient: the idea

- Build G up by **ear decomposition**: start from a cycle, repeatedly attach an *ear* (a path between two existing vertices, all interior vertices new).
- Every bridgeless graph decomposes this way.
  Reference: Bondy & Murty, *Graph Theory*, Ch. 5 §5.3.
- Base case: the starting cycle gets a nowhere-zero flow trivially, orient it consistently and give every edge the value 1.

Adding an ear, in detail

- Attaching ear P (endpoints u, v, all-new interior) creates a new cycle: P together with an existing path Q from u to v.
- Push extra flow c around this whole cycle: every edge of P is brand new, so it simply becomes c; every edge of Q changes from its old value to (old value) $\pm c$.
- An edge of Q is zeroed out only if c exactly cancels its old value, one specific value of c per edge of Q. Since Q is finite, only finitely many values of c are "bad".
- So choose c larger in absolute value than every value used so far, say $c = M + 1$ where M is the current maximum. No existing edge can be cancelled by a c that large.
- Repeat over all ears. The graph is finite, so this terminates, leaving an integer-valued nowhere-zero flow with all values in $\{-M, \ldots, -1, 1, \ldots, M\}$ for some final M.
- Reduce every value mod $(2M + 1)$: conservation is additive, so it survives reduction; and every value already has absolute value $\leq M < 2M + 1$, so none becomes 0. This gives a nowhere-zero $(2M + 1)$-flow.

The Petersen graph: why minimum k is hard



- Cubic, bridgeless, 3-edge-connected: bridgeless gives existence of some flow; cubic and 3-edge-connected is exactly the reduced case Seymour's theorem targets next.
- Has a nowhere-zero 5-flow, consistent with **Tutte's 5-flow conjecture**: every bridgeless graph has a nowhere-zero 5-flow (still open).
- Has **no** nowhere-zero 4-flow. Theorem (Tutte): for cubic graphs, a nowhere-zero 4-flow exists iff the graph is properly 3-edge-colourable, via the three nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The Petersen graph is a *snark*: not 3-edge-colourable.

This is why Tutte's 4-flow conjecture has to exclude graphs with a Petersen minor. The best known unconditional bound, next, is Seymour's.

Seymour's 6-flow theorem

Theorem (Seymour, 1981)

Every bridgeless graph has a nowhere-zero 6-flow.

- Best known *unconditional* bound: no extra hypotheses, just bridgeless.
- Sits between the easy existence argument above (which gives some large, graph-size-dependent k) and Tutte's still-open **5-flow conjecture**.

Why it reduces to cubic, 3-edge-connected graphs

- **Vertex splitting:** any vertex of degree > 3 can be replaced by a small tree of degree-3 vertices without changing whether a nowhere-zero flow exists; a flow on the split graph glues back to one on the original.
- Repeating this turns any graph into a cubic one.
- A 2-edge-cut can be handled by solving each side separately and gluing the flows back together, so it suffices to assume 3-edge-connectivity too.
- So it is enough to prove: every 3-edge-connected cubic graph has a nowhere-zero 6-flow. Standard move in the area: reduce to a well-behaved minimal case, then solve that case.

Vertex splitting: picture 2-edge-cut: picture

Where the 6 comes from: the flavour

Note

What follows is the *shape* of the argument, not the real proof. The full construction is genuinely technical.

- Build two flows on the *same* graph: a $\mathbb{Z}_3$-flow f_1 and a $\mathbb{Z}_2$-flow f_2, chosen so that no edge is zero under both at once.
- This is *not* obvious, and finding such a compatible pair is the real technical heart of the proof; we are only naming what the pair achieves, not how it is built.
- Pair them up: $(f_1(e), f_2(e)) \in \mathbb{Z}_3 \times \mathbb{Z}_2$ for each edge e.
- Since $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$, and no edge is the identity in both coordinates at once, this pairing is a nowhere-zero $\mathbb{Z}_6$-flow.
- So 6 is not an arbitrary constant: it falls out of combining a 2-flow-type and 3-flow-type structure.
- Proof technique: minimal counterexample, the same broad style as the reducibility arguments (Birkhoff, Appel–Haken, Robertson et al.) in the Four Colour Theorem proof.

Back to the Tutte polynomial

Recall from July: $T(G; x, y)$, the universal invariant.

Chromatic polynomial (colourings, $y = 0$)

$$P(G; k) = (-1)^{r(G)} k^{c(G)} T(G; 1 - k, 0)$$

Flow polynomial (flows, $x = 0$)

$$F(G; k) = (-1)^{|E|-|V|+1} T(G; 0, 1 - k)$$

Two evaluations of the *one* polynomial: colourings along $y = 0$, flows along $x = 0$. The same tool from July does double duty here, and $F(G; k)$ is exactly the polynomial that counts nowhere-zero Γ-flows for $|\Gamma| = k$, the fact flagged earlier.

The Cycle Double Cover Conjecture

Conjecture (Szekeres 1973, Seymour 1979)

Every bridgeless graph has a list of cycles such that every edge lies in exactly two of them.

- Flagged in July as breaking news: OpenAI announced a claimed proof in July 2026.
- Independently written up by Sang-Il Oum (arXiv:2607.16356), through several revisions.
- **Caveat:** as of writing, not yet appeared in a peer-reviewed journal. Recent and closely scrutinised, not yet textbook-settled. Worth checking again nearer the time.

How the proof uses today's material

- **Reduce to cubic graphs.** Classical tool (Bondy & Murty, Thm 5.5): the conjecture is true in general if and only if it is true for all nonseparable cubic graphs.

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- Same reduction philosophy as the Four Colour proof: cut down to a well-behaved minimal case, then solve that case.

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A matroid coda

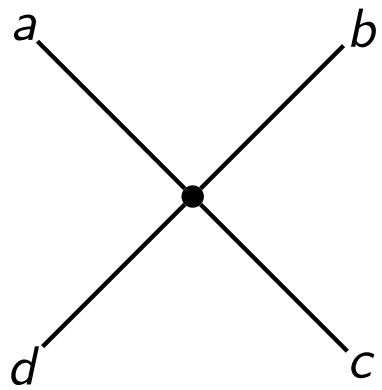
- A flow is an assignment balanced at every vertex; the set of all such assignments is the **cycle space** of the graph, so that is where our flow lives.
- "Balanced at every vertex" is the same as "balanced around every cocircuit": a vertex's incident edges are one example of a cocircuit, and vertex-stars span the whole **cocycle space**.
- Abstract matroids have no vertices, but they always have a dual, hence always have cocircuits. So this is the route by which flows generalise beyond graphs: restate conservation cocircuit by cocircuit, and it survives the trip.
- Maps, matroids, and now a proof: the thread from July continues.

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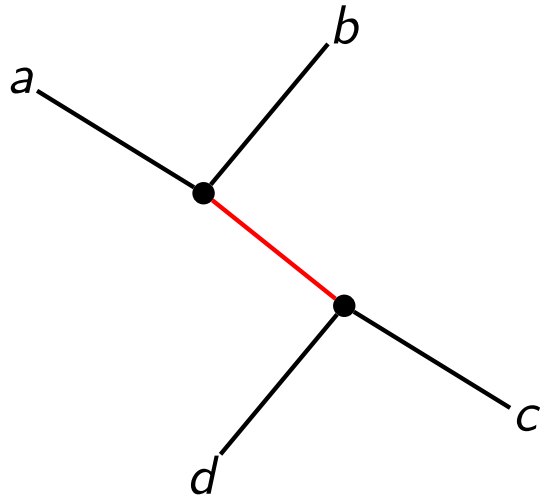
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Backup slides

Backup: vertex splitting, a picture



degree-4 vertex v



two degree-3 vertices, joined by a new edge

v's four edges are shared out, two to v_1 and two to v_2, and one new edge joins them. A flow on the right glues back to one on the left: give the new (red) edge the value forced by conservation at v_1, and conservation at v_2 then follows automatically. Degree-d vertices split the same way into $d-2$ new degree-3 vertices in a path. ← Back to main talk

Backup: handling a 2-edge-cut

- Suppose $\{e_1, e_2\}$ is a 2-edge-cut, splitting G into pieces G_1 and G_2, each containing one end of e_1 and one end of e_2.
- Conservation forces $f(e_1) = \pm f(e_2)$ (whatever crosses out of G_1 via e_1 must return via e_2), so the pair (e_1, e_2) behaves like a single edge joining G_1 and G_2.
- So: contract e_1, e_2 to one edge in each of G_1 and G_2 separately, find a nowhere-zero flow on each smaller piece, then split that single edge's value back across e_1 and e_2 to reassemble a flow on G.
- This lets an induction on graph size handle any 2-edge-cut, so it is enough to also assume 3-edge-connectivity.

← Back to main talk

Backup: from a flow to a cover, the mechanism

Note

Again the *shape* of the argument, not the full proof.

- Start from a nowhere-zero Γ-flow f (e.g. $\Gamma = \mathbb{Z}_2^3$, order 8). To each edge e attach a 2-element subset $P_e \subset \Gamma$, built from $f(e)$.
- For each $s \in \Gamma$, let $M_s = \{e : s \in P_e\}$.
- Conservation makes every vertex have even degree in M_s, so M_s is a disjoint union of cycles.
- Each P_e has exactly two elements, so each edge e lies in exactly two of the sets M_s.
- Collect the cycle components of every M_s, over all $s \in \Gamma$: every edge appears in exactly two of them. That list of cycles is the cycle double cover.

← Back to main talk