Chicks, Eggs and Advertising

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Some time ago I taught Mathematics in a department of Business. I was quite surprised that most students were so afraid of Maths that they weren't even interested whether or not their enterprise was profitable or not. Even some of my colleagues, especially those who taught Marketing, were very shy of anything to do with Maths. I began to wonder if there were any good applications of Mathematics to Advertising. A classic problem in probability is usually phrased in term of chicks and eggs:

My hen lays eggs according to a Poisson distribution with mean λ. The probability that she lays n eggs tomorrow is $p(n) = \lambda^n e^{-\lambda}/n!$. The eggs hatch, or not, according to a binomial probability, with p as the probability of an egg hatching into a chick. What will be the distribution of chicks?

Instead of eggs and chicks suppose we consider enquiries and orders. Suppose my company make a product that is large and expensive so that customers will either order one or none. The adverts my company buys generate enquiries, these could be phone calls, letters or most probably e-mails. Assume the enquiries can be modelled by a Poisson distribution. The mean number of enquiries λ, is now some measure of the effectiveness of the advertising. In the enquiries potential customers (also known as "punters"), want to know the cost, delivery arrangement and so forth. On the basis of this information they may place an order. Whether or not they place an order can be modelled by a binomial distribution. Clearly the company would like to know about the distribution of orders.

Another story we could tell about the same underlying problem concerns internet shopping. Companies spend money advertising their websites, but many people who visit websites may go through most of the process of ordering items but then leave the site without completing the order. I have seen figures quoted that something like 70% of visitors to some websites bail before completing their order.

The problem, however we state it, concerns the convolution of a Poisson distribution with a Binomial distribution. To get k orders we might have k enquiries and they all turn into orders, or we could have $k+1$ enquiries but only k turn into orders and so on. So if we write $\rho(k)$ as the probability of getting k orders, then

$$\rho(k) = \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \binom{i}{k} p^k q^{i-k}$$

where $q = (1 - p)$, the probability the enquiry fails to turn into an order. To sum the series it is convenient to make a change of index variable so that the sum runs from 0 to ∞. Let $j = i - k$, so that $i = k + j$. Now the probability can be written,

$$\rho(k) = \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \binom{k+j}{k} p^k q^j.$$

Taking out the constants and expanding the binomial coefficient gives,

$$\rho(k) = \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda q)^j}{j!} = \frac{(\lambda p)^k}{k!} e^{-\lambda} e^{\lambda q}.$$

Remembering that $q = 1 - p$ gives,

$$\rho(k) = \frac{(\lambda p)^k}{k!} e^{-(\lambda p)}.$$

This is a Poisson distribution again, but now with mean λp.

With different advertising we may be able to increase λ, but this isn't necessarily a good thing. More enquiries from people who are not really interested in the product and are not going to place an order just waste the company's time, (this is why such punters are known as "time wasters"). Really, we would like to increase p, this can be done by more targeted advertising, but this would probably decrease λ. That is, we would expect targeting of our advertising to generate fewer enquiries but with a higher chance that each will turn into an order. Clearly, it is the product λp that we should attempt to increase for more orders. This could be a risky strategy. The variance of the Poisson distribution is the same as its mean, so increasing λp will also increase the variance.

These ideas can be pushed a little further by relaxing the the assumption that customers only buy one or no items. To keep things simple though, we assume that customers buy a discrete number of the same item or spend integer multiples of some basic amount. Let's use the variable k now to count the number of items ordered. We can assume that the probability that $k = 0$ is about 70%, but what distribution should we use for k? I don't have a good answer for this, really I should go and find some data to see what the distribution looks like. For the purposes of this article I will just assume the spending for an individual punter is distributed according to a geometric distribution. So the probability of ordering k items is given by the probability mass function,

$$g(k) = pq^k, \qquad \text{where} \quad q = 1 - p,$$

and from the data $g(0) = p \approx 0.7$. Next, suppose that we have persuaded, by advertising, n punters to visit our website, what is the distribution giving the probability that they will collectively buy κ items? This convolution can be found by multiplying the generating functions for the individual geometric distributions. The generating function for the geometric distribution is,

$$G(z) = \sum_{k=0}^{\infty} pq^k z^k = p(1 - qz)^{-1},$$

since this is the sum of a geometric series. For n punters, the generating function of the distribution is the nth power of the above,

$$G^n(z) = p^n (1 - qz)^{-n}.$$

The probability that the n punters collectively order κ items is then the coefficient of z^κ in the series expansion of $G^n(z)$. Using the binomial theorem to expand the bracket gives,

$$G^n(z) = p^n(1 - qz)^{-n} = p^n \sum_{\kappa=0}^{\infty} \binom{\kappa + n - 1}{\kappa} (qz)^\kappa.$$

So we can see that the probability that n punters order κ items is distributed according to a negative binomial distribution, with probability mass function,

$$nb(n,\, \kappa) = \binom{\kappa + n - 1}{\kappa} p^n q^\kappa.$$

As before, the probability that the advertising lays n punters is assumed to follow a Poisson distribution. The probability that we get orders for κ items is then the convolution,

$$f(\kappa) = \sum_{n=0}^{\infty} nb(n,\, \kappa) p(n) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \binom{\kappa + n - 1}{\kappa} p^n q^\kappa = e^{-\lambda} q^\kappa \sum_{n=0}^{\infty} \binom{\kappa + n - 1}{\kappa} \frac{(\lambda p)^n}{n!}.$$

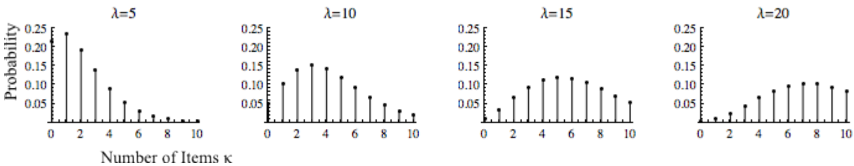

Figure 1: Probability of getting orders for 0 to 10 items. In each case $p = 0.7$, the value of λ varies from $\lambda = 5$ to $\lambda = 20$.

I have no idea how to evaluate this infinite sum, but luckily *Mathematica* does; the result is,

$$\sum_{n=0}^{\infty} \binom{\kappa + n - 1}{\kappa} \frac{(\lambda p)^n}{n!} = (\lambda p)\, {}_1F_1(\kappa + 1; 2; (\lambda p)).$$

The function ${}_1F_1(a; b; x)$ is the confluent hypergeometric function, also called Kummer's hypergeometric function, see the Wikipedia article for more details, [1]. The probability that we get an order for κ items is thus,

$$f(\kappa) = e^{-\lambda} \lambda p q^{\kappa}\, {}_1F_1(\kappa + 1; 2; (\lambda p)).$$

When κ is an integer the confluent hypergeometric function is, in fact, an elementary function. The first few are,

$$\begin{aligned}
{}_1F_1(1; 2; x) &= \frac{1}{x}(e^x - 1), \\
{}_1F_1(2; 2; x) &= e^x, \\
{}_1F_1(3; 2; x) &= \frac{1}{2}(x + 2)e^x, \\
{}_1F_1(4; 2; x) &= \frac{1}{6}(x^2 + 6x + 6)e^x, \\
{}_1F_1(5; 2; x) &= \frac{1}{24}(x^3 + 12x^2 + 36x + 24)e^x,
\end{aligned}$$

and so forth. Figure 1 shows the probabilities of getting orders for 0 to 10 items when $p = 0.7$ and $\lambda = 5$, 10, 15 and 20.

To compute the mean and variance of the distribution it is useful to find the generating function of the distribution $f(\kappa)$. That is we seek,

$$F(z) = \sum_{\kappa=0}^{\infty} f(\kappa) z^{\kappa}.$$

To do this we will take a step back and look at the two variable generating function,

$$\Gamma(s,\, z) = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{\infty} p(n) s^n nb(n,\kappa) z^{\kappa}.$$

This will give the generating function we are after when $s = 1$, $F(z) = \Gamma(1,\, z)$. Substituting for the Poisson and negative binomial distributions gives,

$$\Gamma(s,\, z) = e^{-\lambda} \sum_{n=0}^{\infty}\sum_{\kappa=0}^{\infty} \frac{(p\lambda s)^n}{n!} \binom{\kappa + n - 1}{\kappa} (qz)^{\kappa}.$$

Performing the sum over κ gives,

$$\Gamma(s,\, z) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(p\lambda s)^n}{n!} (1 - qz)^{-n}.$$

The sum over n is just an exponential so we have the result,

$$\Gamma(s,\, z) = e^{-\lambda} e^{\left(\frac{p\lambda s}{1-qz}\right)}.$$

Finally the generating function for $f(\kappa)$ is,

$$F(z) = \Gamma(1,\, z) = e^{-\lambda} e^{\left(\frac{p\lambda}{1-qz}\right)}.$$

The mean and variance of the distribution are now easy to compute,

$$\bar{\kappa} = \frac{dF(1)}{dz} = \frac{\lambda q}{p}$$

and

$$\text{Var} = \frac{d^2F(1)}{dz^2} + \frac{dF(1)}{dz} - \left(\frac{dF(1)}{dz}\right)^2 = \frac{\lambda q(q+1)}{p^2}.$$

In conclusion, there are several ways these ideas could be extended, although one should first check that the notions explored above have some validity! This all assumes that the purpose of advertising is to increase sales but I don't think this is always the case.

Since the 1960s people have been advocating the use of Maths in Marketing, most of the proposed models are extremely simple. There are many articles and blog posts claiming that "There is Math in Marketing...I swear", but the Math(s) referred to is often just calculating a ratio of two numbers to produce a performance measure for an advertising campaign. The deepest application I have come across was a model of Marketing in a duopoly based on differential games, in particular Lanchester's equations, see [2] and references therein. These models were originally developed to model warfare so the idea here is to imaging two companies fighting an advertising war for market share.

In this "Age of Mathematics" I sure that many more example will be developed and, perhaps, used.

References

[1] Wikipedia,"Confluent hypergeometric function". https://en.wikipedia.org/wiki/Confluent_hypergeometric_function

[2] Jørgensen S., Sigué S. (2020) A Lanchester-Type Dynamic Game of Advertising and Pricing. In: Pineau PO., Sigué S., Taboubi S. (eds) Games in Management Science. International Series in Operations Research & Management Science, vol 280. Springer, Cham.