Minors and Tutte invariants for alternating dimaps

Graham Farr

Faculty of I.T., Clayton campus
Monash University
Graham.Farr@monash.edu

Work done partly at: Isaac Newton Institute for Mathematical Sciences (Combinatorics and Statistical Mechanics Programme), Cambridge, 2008; University of Melbourne (sabbatical), 2011; and Queen Mary, University of London, 2011.

30 June 2015
Contraction and Deletion

\[
G
\]

\[
G/e
\]

\[
G/e
\]

\[
G \setminus e
\]

\[
G \setminus e
\]

\[
u = v
\]
Minors

$H$ is a **minor** of $G$ if it can be obtained from $G$ by some sequence of deletions and/or contractions.

The order doesn’t matter. Deletion and contraction **commute**:

\[
G/e/f = G/f/e \\
G \setminus e \setminus f = G \setminus f \setminus e \\
G/e \setminus f = G \setminus f / e \\
G/e \setminus f = G \setminus f / e
\]
Minors

$H$ is a **minor** of $G$ if it can be obtained from $G$ by some sequence of deletions and/or contractions.

The order doesn’t matter. Deletion and contraction **commute**:

\[
\begin{align*}
G/e/f & = G/f/e \\
G \setminus e \setminus f & = G \setminus f \setminus e \\
G/e \setminus f & = G \setminus f / e
\end{align*}
\]

Importance of minors:
- **excluded minor characterisations**
  - planar graphs (Kuratowski, 1930; Wagner, 1937)
  - graphs, among matroids (Tutte, PhD thesis, 1948)
- **counting**
  - Tutte-Whitney polynomial family
Duality and minors

Classical duality for embedded graphs:

\[ G \leftrightarrow G^* \]

vertices ↔ faces
Duality and minors

Classical duality for embedded graphs:

\[ G \leftrightarrow G^* \]

vertices \(\leftrightarrow\) faces

contraction \(\leftrightarrow\) deletion

\[ (G/e)^* = G^* \setminus e \]
\[ (G \setminus e)^* = G^* / e \]
Duality and minors
Duality and minors
Duality and minors

\[ G \xrightarrow{e} G^* \]

\[ G/e \xrightarrow{G^*} G^*/e \]
Duality and minors

\[ G \leftrightarrow G^* \]

\[ G/e \leftrightarrow G^*/e \]
Loops and coloops

- Loop
- Coloop = Bridge = Isthmus
Loops and coloops

loop

coloop = bridge = isthmus

duality
H. E. Dudeney, Puzzling Times at Solvamhall Castle: Lady Isabel's Casket, *London Magazine* 7 (42) (Jan 1902) 584
THE CANTERBURY PUZZLES
AND OTHER CURIOUS PROBLEMS

BY
HENRY ERNEST DUDENEY
AUTHOR OF
"AMUSEMENTS IN MATHEMATICS," ETC.

THE DISSECTION OF RECTANGLES INTO SQUARES

BY R. L. BROOKS, C. A. B. SMITH, A. H. STONE AND W. T. TUTTE

Introduction. We consider the problem of dividing a rectangle into a finite number of non-overlapping squares, no two of which are equal. A dissection of a rectangle \( R \) into a finite number \( n \) of non-overlapping squares is called a squaring of \( R \) of order \( n \); and the \( n \) squares are the elements of the dissection. The term "elements" is also used for the lengths of the sides of the elements. If there is more than one element and the elements are all unequal, the squaring is called perfect, and \( R \) is a perfect rectangle. (We use \( R \) to denote both a rectangle and a particular squaring of it.) Examples of perfect rectangles have been published in the literature.\(^1\)

Our main results are:

Every squared rectangle has commensurable sides and elements.\(^2\) (This is (2.14) below.)

Conversely, every rectangle with commensurable sides is perfectible in an infinity of essentially different ways. (This is (9.45) below.) \(\text{Added in proof.} \) Another proof of this theorem has since been published by R. Sprague: Journal für Mathematik, vol. 182(1940), pp. 60–64; Mathematische Zeitschrift, vol. 46(1940), pp. 460–471.)

In particular, we give in §8.3 a perfect dissection of a square into 26 elements.\(^3\)

There are no perfect rectangles of order less than 9, and exactly two of order 9.\(^4\) (This is (5.23) below.)

from a design for a proposed memorial to Tutte in Newmarket, UK.

https://www.facebook.com/billtutte
THE DISSECTION OF EQUILATERAL TRIANGLES INTO EQUILATERAL TRIANGLES

BY W. T. TUTTE

Received 10 December 1947

1. Introduction

In a previous joint paper (‘The dissection of rectangles into squares’, by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, Duke Math. J. 7 (1940), 312–40), hereafter referred to as (A) for brevity, it was shown that it is possible to dissect a square into smaller unequal squares in an infinite number of ways. The basis of the theory was the association with any rectangle or square dissected into squares of an electrical network obeying Kirchhoff’s laws. The present paper is concerned with the similar problem of dissecting a figure into equilateral triangles. We make use of an analogue of the electrical network in which the ‘currents’ obey laws similar to but not identical with those of Kirchhoff. As a generalization of topological duality in the sphere we find that these networks occur in triplets of ‘trial networks’ $N^1$, $N^2$, $N^3$. We find that it is impossible to dissect a triangle into unequal equilateral triangles but that a dissection is possible into triangles and rhombuses so that no two of these figures have equal sides. Most of the theorems of paper (A) are special cases of those proved below.
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
triad of alternating dimaps
triad of alternating dimaps
triad of alternating dimaps bicubic map
triad of alternating dimaps bicubic map
bicubic map
bicubic map

triad of alternating dimaps bicubic map
Alternating dimaps

*Alternating dimap* (Tutte, 1948):

- directed graph without isolated vertices,
- 2-cell embedded in a disjoint union of orientable 2-manifolds,
- each vertex has even degree,
- \( \forall v \): edges incident with \( v \) are directed alternately into, and out of, \( v \) (as you go around \( v \)).

Genus \( \gamma(G) \) of an alternating dimap \( G \):

\[
V - E + F = 2(k(G) - \gamma(G))
\]
Alternating dimaps

*Alternating dimap* (Tutte, 1948):

- directed graph without isolated vertices,
- 2-cell embedded in a disjoint union of orientable 2-manifolds,
- each vertex has even degree,
- $\forall v$: edges incident with $v$ are directed alternately into, and out of, $v$ (as you go around $v$).

So vertices look like this:
Alternating dimaps

*Alternating dimap* (Tutte, 1948):

- directed graph without isolated vertices,
- 2-cell embedded in a disjoint union of orientable 2-manifolds,
- each vertex has even degree,
- \( \forall v \): edges incident with \( v \) are directed alternately into, and out of, \( v \) (as you go around \( v \)).

So vertices look like this:

\[
\text{Genus } \gamma(G) \text{ of an alternating dimap } G:\n\]

\[
v - E + F = 2(k(G) - \gamma(G))
\]
Alternating dimaps

Three special partitions of $E(G)$:

- clockwise faces
- anticlockwise faces
- in-stars

(An in-star is the set of all edges going into some vertex.)
Three special partitions of $E(G)$:

- clockwise faces
- anticlockwise faces
- in-stars

(An in-star is the set of all edges going into some vertex.) Each defines a permutation of $E(G)$. 

Alternating dimaps
Alternating dimaps

Three special partitions of $E(G)$:
- *clockwise faces* $\sigma_c$
- *anticlockwise faces* $\sigma_a$
- *in-stars* $\sigma_i$

(An *in-star* is the set of all edges going into some vertex.)
Each defines a permutation of $E(G)$. 
Alternating dimaps

Three special partitions of $E(G)$:

- **clockwise faces** $\sigma_c$
- **anticlockwise faces** $\sigma_a$
- **in-stars** $\sigma_i$

(An *in-star* is the set of all edges going into some vertex.)

Each defines a permutation of $E(G)$. These permutations satisfy

$$\sigma_i\sigma_c\sigma_a = 1$$
Triality (Trinity)

Construction of trial map:

clockwise faces $\rightarrow$ vertices $\rightarrow$ anticlockwise faces $\rightarrow$ clockwise faces
Triality (Trinity)

Construction of trial map:

clockwise faces $\rightarrow$ vertices $\rightarrow$ anticlockwise faces $\rightarrow$ clockwise faces

$(\sigma_i, \sigma_c, \sigma_a) \mapsto (\sigma_c, \sigma_a, \sigma_i)$
Triality (Trinity)

Construction of trial map:

clockwise faces $\rightarrow$ vertices $\rightarrow$ anticlockwise faces $\rightarrow$ clockwise faces

$$(\sigma_i, \sigma_c, \sigma_a) \mapsto (\sigma_c, \sigma_a, \sigma_i)$$
Triality (Trinity)

Construction of trial map:

clockwise faces $\rightarrow$ vertices $\rightarrow$ anticlockwise faces $\rightarrow$ clockwise faces

$$(\sigma_i, \sigma_c, \sigma_a) \mapsto (\sigma_c, \sigma_a, \sigma_i)$$
Minor operations

\[ G \]

\[ u \]

\[ v \]

\[ w_1 \]

\[ w_2 \]
Minor operations

\[ G[1]e \]
Minor operations
Minor operations

$G[\omega]e$
Minor operations

$G$

$u$

$e$

$v$

$w_1$

$w_2$
Minor operations

$G[\omega^2]e$
Minor operations

\[ G \]

\[ u \]

\[ e \]

\[ v \]

\[ w_1 \]

\[ w_2 \]
Minor operations

$G$

$u$

$e$

$v$

$e^\omega$

$w_1$

$w_2$
Minor operations

\[ G[1]e \]
(G[1]e)^ω = G^ω[ω^2]e^ω
Minor operations

\[
G^\omega [1] e^\omega = (G[\omega] e)^\omega, \\
G^\omega [\omega] e^\omega = (G[\omega^2] e)^\omega, \\
G^\omega [\omega^2] e^\omega = (G[1] e)^\omega,
\]

\[
G^{\omega^2} [1] e^{\omega^2} = (G[\omega^2] e)^{\omega^2}, \\
G^{\omega^2} [\omega] e^{\omega^2} = (G[1] e)^{\omega^2}, \\
G^{\omega^2} [\omega^2] e^{\omega^2} = (G[\omega] e)^{\omega^2}.
\]

**Theorem**

*If* \( e \in E(G) \) *and* \( \mu, \nu \in \{1, \omega, \omega^2\} \) *then*

\[
G^\mu [\nu] e^\mu = (G[\mu \nu] e)^\mu.
\]

Same pattern as established for other generalised minor operations (GF, 2008/2013...).
Minor operations

$G \rightarrow G^\omega \rightarrow G^{\omega^2}$
Minors: bicubic maps
Minors: bicubic maps

Minors: bicubic maps

Minors: bicubic maps

Relationships

triangulated triangle

\[ \uparrow \downarrow \]

alternating dimaps

\[ \uparrow \downarrow \]

bicubic map (reduction: Tutte 1975)

\[ \uparrow \downarrow \] duality

Eulerian triangulation
Relationships

- triangulated triangle

- alternating dimaps

- bicubic map \textit{(reduction: Tutte 1975)}

- duality

Eulerian triangulation \textit{(reduction, in inverse form \ldots: Batagelj, 1989)}
Relationships

triangulated triangle

\[
\uparrow
\]

alternating dimaps

\[
\downarrow
\]

bicubic map (reduction: Tutte 1975)

\[
\uparrow
\]
duality

Eulerian triangulation (reduction, in inverse form . . . : Batagelj, 1989)

\[
\uparrow
\]
(Cavenagh & Lisoněck, 2008)

spherical latin bitrade
Ultraloops, triloops, semiloops
Ultraloops, triloops, semiloops
Ultraloops, triloops, semiloops

\(\omega\)-loop

ultraloop

1-loop
Ultraloops, triloops, semiloops

\[ \omega \text{-loop} \]

\[ \omega^2 \text{-loop} \]

\[ \text{ultraloop} \]

\[ 1\text{-loop} \]
Ultraloops, triloops, semiloops

ω-loop

ultraloop

ω²-loop

1-loop
Ultraloops, triloops, semiloops
Ultraloops, triloops, semiloops

\[ \omega \text{-loop} \]

\[ \omega^2 \text{-loop} \]
Ultraloops, triloops, semiloops

1-semiloop

ω-loop

ω²-loop

ultraloop

1-loop
Ultraloops, triloops, semiloops

1-semiloop

ω-loop

ω²-loop

ultraloop

1-loop

ω-semiloop
Ultraloops, triloops, semiloops

- 1-loop
- ω-loop
- ω²-loop
- ω-semiloop
- 1-semiloop
- ω²-semiloop
- ultraloop
Ultraloops, triloops, semiloops

1-semiloop

ω-loop

ultraloop

ω²-loop

1-loop

ω²-semiloop

ω-semiloop
Ultraloops, triloops, semiloops

1-semiloop

ω-loop

ω²-loop

ultraloop

1-loop

ω²-semiloop

ω-semiloop
Ultraloops, triloops, semiloops

- Ultraloop
- 1-loop
- $\omega$-loop
- $\omega^2$-loop
- 1-semiloop
- $\omega$-semiloop
- $\omega^2$-semiloop
Ultraloops, triloops, semiloops: the bicubic map

trihedron
(ultraloop)
Ultraloops, triloops, semiloops: the bicubic map

trihedron (ultraloop)

digon (triloop)
Ultraloops, triloops, semiloops: the bicubic map

trihedron (ultraloop)

digon (triloop)

(semiloop)
Non-commutativity

Some bad news: sometimes,

\[ G[\mu]e[\nu]f \neq G[\nu]f[\mu]e \]
$G[\omega]f[1]e$
Theorem
Except for the above situation and its trials, reductions commute.

Corollary
If $\mu = \nu$, or one of $e, f$ is a triloop, then reductions commute.

Theorem
Except for the above situation and its trials, reductions commute.


Corollary
If \( \mu = \nu \), or one of \( e, f \) is a triloop, then reductions commute.
Theorem

All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$. 

Trimestone graph

$G$

$u$

$e$

$v$

$w_1$

$w_2$
Theorem

All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$. 
Theorem
All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$. 
Theorem

All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$. 

Trimedial graph

$G$

$\text{tri}(G)$
Theorem

All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$.
Theorem
All pairs of reductions on $G$ commute if and only if the triloops of $G$ form a vertex cover in $\text{tri}(G)$.
Non-commutativity

Theorem

All sequences of reductions on $G$ commute if and only if each component of $G$ has the form \ldots
Non-commutativity

Theorem

All sequences of reductions on G commute if and only if each component of G has the form . . .
Non-commutativity

**Problem**
Characterise alternating dimaps such that all pairs of reductions commute *up to isomorphism*:

\[ \forall \mu, \nu, e, f : \quad G[\mu]f[\nu]e \cong G[\nu]e[\mu]f \]
Excluded minors for bounded genus

$k$-posy:
An alternating dimap with ...
  ▶ one vertex,
  ▶ $2k + 1$ edges,
  ▶ two faces.

\[ V - E + F = 1 - (2k + 1) + 2 = 2 - 2k \]

Genus of $k$-posy $= k$

Theorem

A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$. 
Excluded minors for bounded genus

\( k\text{-posy} \):
An alternating dimap with . . .

- one vertex,
- \(2k + 1\) edges,
- two faces.

\[ V - E + F = 1 - (2k + 1) + 2 = 2 - 2k \]

Genus of \( k\)-posy = \( k \)

Theorem
A nonempty alternating dimap \( G \) has genus < \( k \) if and only if none of its minors is a disjoint union of posies of total genus \( k \).

Excluded minors for bounded genus

0-posy:
Excluded minors for bounded genus

0-posy:
Excluded minors for bounded genus

1-posy:
Excluded minors for bounded genus

2-posy: first:
Excluded minors for bounded genus

2-posy:    second:

\begin{figure}
\centering
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (90:3) {$b$};
  \node (c) at (180:3) {$c$};
  \node (d) at (270:3) {$d$};
  \node (e) at (0:3) {$e$};
  \draw[->] (a) to (b);
  \draw[->] (a) to (c);
  \draw[->] (a) to (d);
  \draw[->] (a) to (e);
  \draw[->] (b) to (c);
  \draw[->] (b) to (d);
  \draw[->] (b) to (e);
  \draw[->] (c) to (d);
  \draw[->] (c) to (e);
  \draw[->] (d) to (e);
\end{tikzpicture}
\end{figure}
Excluded minors for bounded genus

2-posy: third:
Theorem

A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$. 
Theorem
A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

Proof.
Theorem

A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

Proof.

$(\implies)$ Easy.
Excluded minors for bounded genus

Theorem
A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

Proof.

$(\implies)$ Easy.

$(\impliedby)$ Show:

$\gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k$. 
Excluded minors for bounded genus

Theorem
A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

Proof.

(⇒) Easy.

(⇐) Show:

$\gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k$.

Induction on $|E(G)|$. 
Theorem
A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

Proof.

($\implies$) Easy.

($\impliedby$) Show:

$\gamma(G) \geq k \implies \exists$ minor $\cong$ disjoint union of posies, total genus $k$.

Induction on $|E(G)|$.

Inductive basis:

$|E(G)| = 1 \implies G$ is an ultraloop $\implies$ 0-posy minor.
Excluded minors for bounded genus

Showing …

\( \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k \).
Excluded minors for bounded genus

Showing ... 

\( \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \)

Inductive step: Suppose true for alt. dimaps of \(< m \) edges.
Excluded minors for bounded genus

Showing . . .

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \( < m \) edges.
Let \( G \) be an alternating dimap with \( |E(G)| = m \).
Excluded minors for bounded genus

Showing . . .

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \(< m \) edges.

Let \( G \) be an alternating dimap with \( |E(G)| = m \).

\( G[1]e, \ G[\omega]e, \ G[\omega^2]e \) each have \( m - 1 \) edges.
Excluded minors for bounded genus

Showing ... 

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \(< m\) edges. Let \( G \) be an alternating dimap with \( |E(G)| = m \).

\[ G[1]e, \ G[\omega]e, \ G[\omega^2]e \] each have \( m - 1 \) edges. 

\[ \therefore \text{ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus ...} \]

\[ \gamma(G[1]e), \ \gamma(G[\omega]e), \ \gamma(G[\omega^2]e), \] respectively.
Excluded minors for bounded genus

Showing ...

\( \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \)

Inductive step: Suppose true for alt. dimaps of \(< m\) edges.
Let \( G \) be an alternating dimap with \(|E(G)| = m\).
\( G[1]e, \quad G[\omega]e, \quad G[\omega^2]e \) each have \( m - 1 \) edges.
\( \therefore \) by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus ...
\( \gamma(G[1]e), \quad \gamma(G[\omega]e), \quad \gamma(G[\omega^2]e), \) respectively.
If any of these = \( \gamma(G) \): done.
Excluded minors for bounded genus

Showing...

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \( < m \) edges.
Let \( G \) be an alternating dimap with \( |E(G)| = m \).

\[ G[1]e, \quad G[\omega]e, \quad G[\omega^2]e \quad \text{each have } m - 1 \text{ edges}. \]

\[ \therefore \text{ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus } \ldots \]
\[ \gamma(G[1]e), \quad \gamma(G[\omega]e), \quad \gamma(G[\omega^2]e), \quad \text{respectively}. \]

If any of these = \( \gamma(G) \): done.

It remains to consider:
\[ \gamma(G[1]e) = \gamma(G[\omega]e) = \gamma(G[\omega^2]e) = \gamma(G) - 1. \]
Excluded minors for bounded genus

Showing . . .

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \(< m\) edges.
Let \(G\) be an alternating dimap with \(|E(G)| = m\).
\(G[1]e, G[\omega]e, G[\omega^2]e\) each have \(m - 1\) edges.
\(\therefore\) by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus . . .
\(\gamma(G[1]e), \gamma(G[\omega]e), \gamma(G[\omega^2]e),\) respectively.
If any of these = \(\gamma(G)\): done.
It remains to consider:
\(\gamma(G[1]e) = \gamma(G[\omega]e) = \gamma(G[\omega^2]e) = \gamma(G) - 1.\)

↑
proper
1-semiloop
Excluded minors for bounded genus

Showing ... 

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \(< m\) edges.
Let \(G\) be an alternating dimap with \(|E(G)| = m\).

\(G[1]e, \quad G[\omega]e, \quad G[\omega^2]e\) each have \(m - 1\) edges.

\[ \therefore \text{ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus ...} \]

\[ \gamma(G[1]e), \quad \gamma(G[\omega]e), \quad \gamma(G[\omega^2]e), \quad \text{respectively.} \]

If any of these = \(\gamma(G)\): done.

It remains to consider:
\[ \gamma(G[1]e) = \gamma(G[\omega]e) = \gamma(G[\omega^2]e) = \gamma(G) - 1. \]

↑
proper
\(\omega^2\)-semiloop
Excluded minors for bounded genus

Showing . . .

\[ \gamma(G) \geq k \implies \exists \text{ minor } \cong \text{ disjoint union of posies, total genus } k. \]

Inductive step: Suppose true for alt. dimaps of \(< m\) edges. Let \(G\) be an alternating dimap with \(|E(G)| = m\).

\(G[1]e, G[\omega]e, G[\omega^2]e\) each have \(m-1\) edges. ∴ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus . . .

\(\gamma(G[1]e), \gamma(G[\omega]e), \gamma(G[\omega^2]e)\), respectively. If any of these = \(\gamma(G)\): done.

It remains to consider:

\(\gamma(G[1]e) = \gamma(G[\omega]e) = \gamma(G[\omega^2]e) = \gamma(G) - 1.\)

↑

proper

\(\omega\)-semiloop
Excluded minors for bounded genus
Excluded minors for bounded genus

$F_1$
Excluded minors for bounded genus
Excluded minors for bounded genus
Excluded minors for bounded genus

\[ F_1 \]

\[ F_2 \]

\[ F_1 \]
Excluded minors for bounded genus
Excluded minors for bounded genus
Excluded minors for bounded genus
Tutte polynomial of a graph (or matroid)

\[ T(G; x, y) = \sum_{X \subseteq E} (x - 1)^{\rho(E) - \rho(X)} (y - 1)^{\rho^*(E) - \rho^*(E \setminus X)} \]

where

\[ \rho(Y) = \text{rank of } Y \]
\[ = (\# \text{vertices that meet } Y) - (\# \text{ components of } Y), \]

\[ \rho^*(Y) = \text{rank of } Y \text{ in the dual, } G^* \]
\[ = |X| + \rho(E \setminus X) - \rho(E). \]
Tutte polynomial of a graph (or matroid)

\[ T(G; x, y) = \sum_{X \subseteq E} (x - 1)^{\rho(E) - \rho(X)} (y - 1)^{\rho^*(E) - \rho^*(E\setminus X)} \]

where

\[ \rho(Y) = \text{rank of } Y \]
\[ = (\text{#vertices that meet } Y) - (\text{# components of } Y), \]

\[ \rho^*(Y) = \text{rank of } Y \text{ in the dual, } G^* \]
\[ = |X| + \rho(E \setminus X) - \rho(E). \]

By appropriate substitutions, it yields:

numbers of colourings, acyclic orientations, spanning trees, spanning subgraphs, forests, . . .
Tutte polynomial of a graph (or matroid)

\[ T(G; x, y) = \sum_{X \subseteq E} (x - 1)^{\rho(E) - \rho(X)} (y - 1)^{\rho^*(E) - \rho^*(E\setminus X)} \]

where

\[ \rho(Y) = \text{rank of } Y = (\# \text{vertices that meet } Y) - (\# \text{ components of } Y), \]

\[ \rho^*(Y) = \text{rank of } Y \text{ in the dual, } G^* = |X| + \rho(E \setminus X) - \rho(E). \]

By appropriate substitutions, it yields:

numbers of colourings, acyclic orientations, spanning trees, spanning subgraphs, forests, \ldots

chromatic polynomial, flow polynomial, reliability polynomial, Ising and Potts model partition functions, weight enumerator of a linear code, Jones polynomial of an alternating link, \ldots
Tutte polynomial of a graph (or matroid)

Deletion-contraction relation:

\[ T(G; x, y) = \begin{cases} 
1, & \text{if } G \text{ is empty,} \\
x \ T(G \setminus e; x, y), & \text{if } e \text{ is a coloop (i.e., bridge),} \\
y \ T(G/e; x, y), & \text{if } e \text{ is a loop,} \\
T(G \setminus e; x, y) + T(G/e; x, y), & \text{if } e \text{ is neither a coloop nor a loop.}
\end{cases} \]
Tutte polynomial of a graph (or matroid)

Deletion-contraction relation:
\[ T(G; x, y) = \begin{cases} 
1, & \text{if } G \text{ is empty}, \\
x \ T(G \setminus e; x, y), & \text{if } e \text{ is a coloop (i.e., bridge)}, \\
y \ T(G/e; x, y), & \text{if } e \text{ is a loop}, \\
T(G \setminus e; x, y) + T(G/e; x, y), & \text{if } e \text{ is neither a coloop nor a loop}. 
\end{cases} \]

Recipe Theorem (in various forms: Tutte, 1948; Brylawski, 1972; Oxley & Welsh, 1979):
If \( F \) is an isomorphism invariant and satisfies \ldots \n\[ F(G) = \begin{cases} 
x \ F(G \setminus e), & \text{if } e \text{ is a coloop (i.e., bridge)}, \\
y \ F(G/e), & \text{if } e \text{ is a loop}, \\
a \ F(G \setminus e) + b \ F(G/e), & \text{if } e \text{ is neither a coloop nor a loop}. 
\end{cases} \]
\ldots then it can be obtained from the Tutte polynomial using appropriate substitutions and factors.
Tutte invariant for alternating dimaps

– an isomorphism invariant $F$ such that:

$$F(G) =$$

$$\begin{cases} 
1, & \text{if } G \text{ is empty}, \\
wf(G - e), & \text{if } e \text{ is an ultraloop}, \\
x F(G[1]e), & \text{if } e \text{ is a proper 1-loop}, \\
y F(G[\omega]e), & \text{if } e \text{ is a proper } \omega\text{-loop}, \\
z F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-loop}, \\
a F(G[1]e) + b F(G[\omega]e) + c F(G[\omega^2]e), & \text{if } e \text{ is not a triloop}. 
\end{cases}$$
Tutte invariant for alternating dimaps

Theorem
The only Tutte invariants of alternating dimaps are:
(a) $F(G) = 0$ for nonempty $G$,
(b) $F(G) = 3|E(G)| a|V(G)| b^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$,
(c) $F(G) = a|V(G)| b^{\text{c-faces}(G)} (-c)^{\text{a-faces}(G)}$,
(d) $F(G) = a|V(G)| (-b)^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$,
(e) $F(G) = (-a)|V(G)| b^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$. 
Extended Tutte invariant for alternating dimaps

– an isomorphism invariant $F$ such that:

$$F(G) =
\begin{cases}
1, & \text{if } G \text{ is empty,} \\
w F(G - e), & \text{if } e \text{ is an ultraloop,} \\
x F(G[1]e), & \text{if } e \text{ is a proper 1-loop,} \\
y F(G[\omega]e), & \text{if } e \text{ is a proper } \omega\text{-loop,} \\
z F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-loop,} \\
a F(G[1]e) + b F(G[\omega]e) + c F(G[\omega^2]e), & \text{if } e \text{ is a proper } 1\text{-semiloop,} \\
d F(G[1]e) + e F(G[\omega]e) + f F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega\text{-semiloop,} \\
g F(G[1]e) + h F(G[\omega]e) + i F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-semiloop,} \\
j F(G[1]e) + k F(G[\omega]e) + l F(G[\omega^2]e), & \text{if } e \text{ is not a triloop.}
\end{cases}$$

Extended Tutte invariant for alternating dimaps

For any alternating dimap $G$, define $T_c(G; x, y)$ and $T_a(G; x, y)$ as follows.

$$T_c(G; x, y) = \begin{cases} 
1, & \text{if } G \text{ is empty}, \\
T_c(G[*]e; x, y), & \text{if } e \text{ is an } \omega^2\text{-loop}; \\
x T_c(G[\omega^2]e; x, y), & \text{if } e \text{ is an } \omega\text{-semiloop}; \\
y T_c(G[1]e; x, y), & \text{if } e \text{ is a proper } 1\text{-semiloop or an } \omega\text{-loop}; \\
T_c(G[1]e; x, y) + T_c(G[\omega^2]e; x, y), & \text{if } e \text{ is not a semiloop.}
\end{cases}$$

$$T_a(G; x, y) = \begin{cases} 
1, & \text{if } G \text{ is empty}, \\
T_a(G[*]e; x, y), & \text{if } e \text{ is an } \omega\text{-loop}; \\
x T_a(G[\omega]e; x, y), & \text{if } e \text{ is an } \omega^2\text{-semiloop}; \\
y T_a(G[1]e; x, y), & \text{if } e \text{ is a proper } 1\text{-semiloop or an } \omega^2\text{-loop}; \\
T_a(G[1]e; x, y) + T_a(G[\omega]e; x, y), & \text{if } e \text{ is not a semiloop.}
\end{cases}$$
Extended Tutte invariant for alternating dimaps

Theorem

For any plane graph $G$,

$$T(G; x, y) = T_c(\text{alt}_c(G); x, y)$$
Theorem

*For any plane graph* $G$, 

$$T(G; x, y) = T_c(\text{alt}_c(G); x, y)$$
Extended Tutte invariant for alternating dimaps

Theorem
For any plane graph $G$,

$$T(G; x, y) = T_c(alt_c(G); x, y)$$
Theorem

For any plane graph $G$,

$$T(G; x, y) = T_c(alt_c(G); x, y) = T_a(alt_a(G); x, y).$$
Theorem

For any plane graph $G$, 

$$T(G; x, y) = T_c(\text{alt}_c(G); x, y) = T_a(\text{alt}_a(G); x, y).$$
Extended Tutte invariant for alternating dimaps

\[ T_i(G; x) = \begin{cases} 
1, & \text{if } G \text{ is empty,} \\
T_i(G[*]e; x), & \text{if } e \text{ is a 1-loop (including an ultraloop);} \\
x T_i(G[\omega^2]e; x), & \text{if } e \text{ is a proper } \omega\text{-semiloop or an } \omega^2\text{-loop;} \\
x T_i(G[\omega]e; x), & \text{if } e \text{ is a proper } \omega^2\text{-semiloop or an } \omega\text{-loop;} \\
T_i(G[\omega]e; x) + T_i(G[\omega^2]e; x), & \text{if } e \text{ is not a semiloop.}
\end{cases} \]
Theorem

For any plane graph $G$, 

$$T(G; x, x) = T_i(alt_i(G); x).$$

Extended Tutte invariant for alternating dimaps

Theorem

For any plane graph $G$,

$$T(G; x, x) = T_i(alt_i(G); x).$$
Extended Tutte invariant for alternating dimaps

Theorem

For any plane graph $G$,

$$T(G; x, x) = T_i(\text{alt}_i(G); x).$$
Extended Tutte invariant for alternating dimaps

Theorem

*For any plane graph* $G$,

$$T(G; x, x) = T_i(\text{alt}_i(G); x).$$
Extended Tutte invariant for alternating dimaps

Theorem

For any plane graph $G$,

$$T(G; x, x) = T_i(alt_i(G); x).$$
Extended Tutte invariant for alternating dimaps

**Theorem**

*For any plane graph $G$,*

\[
T(G; x, x) = T_i(alt_i(G); x).
\]
References

For more information:

For more information:

References

For more information:


For less information:
For more information:


For less information: