A Theorem of Euler's: The Penatagonal Number Theorem - another proof from The BOOK

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That busy bloke Euler observed that astonishingly:

$$\prod_{k \geq 1} (1 - x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \dots$$

Those familiar with them will have identified the exponents as the first few *Generalised Pentagonal numbers* which have the form:

$$f_n = \frac{3n^2 + n}{2}$$

for $n = 0, \pm 1, \pm 2, \dots$

If this is really true, how do we prove it?

Stepping back, another of Euler's investigations concerned the infinite product:

$$\prod_{k\geq 1}(1+x^k) = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \dots$$

In this product is it is easy to identify the coefficient of x^n as the number $d(n)$ of partitions of n into distinct parts. So for instance 4 has only 2 such partitions: 4 itself, and $1, 3$ whereas 5 has the 3 distinct partitions: 5 $1, 4$,$3, 2$. But does this help us? No, but what about partitions where the parts are not distinct?

The corresponding formal product for partitions into non-distinct parts is:

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)\dots(1+x^k+x^{2k}+x^{3k}+\dots)\dots$$

It is easy to see that if everything was multiplied out, the coefficient of x^n would count one for each partition of n, this time allowing repeats. The first bracket above provides one of each summand, the second two, etc

But obviously each bracket in the above product is a geometric series, and so it can be written as:

$$(1-x)^{-1}(1-x^2)^{-1}\dots(1-x^k)^{-1}\dots = \prod_{k\geq 1}(1-x^k)^{-1}$$

So denoting by p_n the number of partitions of n for $n \geq 0$ allowing repeats, we can write the expansion of the above as:

$$\prod_{k\geq 1}(1-x^k)^{-1} = \sum_{n\geq 0} c_n x^n$$

But this is the exact inverse of the product we are trying to evaluate! i.e.

$$\prod_{k\geq 1}(1-x^k)\prod_{k\geq 1}(1-x^k)^{-1}=1$$

To recap, if we write the expansion of $\prod_{k\geq 1}(1-x^k)$ as

$$\prod_{k\geq 1}(1-x^k)=\sum_{n\geq 0}c_n x^n$$

then we suspect that if n is of the form

$$n=\frac{3j^2+j}{2};\ j=\ 0,\pm 1,\pm 2,\ldots$$

then $c_n=1$ if j is even, and $c_n=-1$ if j is odd.

So multiplying out the above identity:

$$(1 + c_1x + c_2x^2 + \dots c_px^p \dots)(1 + p_1x + p_2x^2 + \dots p_qx^q + \dots) = 1$$

we get, obviously:

$$\sum_{0 \leq k \leq n} c_k p(n-k) = 0$$

Remember that we know what the p_n are, and also that $c_0 = 1$, so in fact this equation (recurrence relation) determines the c_k, we just need to know what it is. So, given that we know what we think it is, we can turn this round and ask instead if the conjectured values satisfy this relation.

So using the symbol $f_j = \frac{3j^2+j}{2}$, $j = 0, \pm 1, \pm 2, \ldots$, and substituting the conjecture values into the relation, we get, for all $n \geq 0$:

$$\sum_{j \text{ even } f_j \leq n} p(n - f_j) = \sum_{j \text{ odd } f_j \leq n} p(n - f_j)$$

So, this is maybe a scary relation, but all we need to ask to prove the conjecture is, is this relation true? To re-frame this, is we denote the set of all partitions of n by $P(n)$, then we want to establish a $1-1$ relation:

$$\phi : \bigcup_{j \text{ even } f_j \leq n} P(n - f_j) \mapsto \bigcup_{j \text{ odd } f_j \leq n} P(n - f_j)$$

The following bijection was devised by David Bressoud and Doron Zeilburger.

Firstly notice that

$$f_{j+1} - f_j = \frac{3(j+1)^2 + j + 1}{2} - \frac{3j^2 + j}{2} = 3j + 2$$

and $f_{j-1} - f_j = -(3j - 1)$ So if a partition of $n - f_j$ is

$$n - f_j = \lambda_1 + \lambda_2 + \ldots + \lambda_t;\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t \geq 1$$

then this is mapped by ϕ to:

$$(t + 3j - 1) + (\lambda_1 - 1) + (\lambda_2 - 1) + \ldots + (\lambda_t - 1) \qquad \textit{if } t + 3j \geq \lambda_1$$

$$(\lambda_2 + 1) + \lambda_3 + 1) + \ldots + (\lambda_t + 1) + 1 + 1 + \ldots + 1 \quad \textit{otherwise}$$

where there are $\lambda_1 - t - 3j - 1$ ones in the second case.

The total of the first mapping is $n - f_j + 3j - 1 = n - f_{j-1}$, and so it is a partition of $n - f_{j-1}$. The second mapping totals to $n - f_j - 3j - 2 = n - f_{j+1}$.
Thus the mapping maps

$$\phi : \bigcup_{j \text{ even } f_j \leq n} P(n - f_j) \leftrightarrow \bigcup_{j \text{ odd } f_j \leq n} P(n - f_j)$$

In fact ϕ is an involution, i.e ϕ^2 is the identity, so is therefore a bijection. This is easily checked, but as an example,if we choose the partition $4 + 3 + 2 + 1$ and $j = -2$, then $t = 4$, $f_j = 5$ and $n = 15$. So $t + 3j = -2 < \lambda_1 = 4$. So the second case applies, and the partition is mapped by ϕ to: $4 + 3 + 2 + 1 + 1 + 1 + 1 + 1$ with $j = -1$.
Applying ϕ again, since we now have $t = 8$ and $\lambda_1 = 4$, we have $t + 3j = 5 > \lambda$, so now the first case applies, and ϕ maps the partition to: $4 + 3 + 2 + 1 + 0 + 0 + 0 + 0 + 0 + 0$. Since we discard the zeroes, we are back where we started.

References

1 D. BRESSOUD & D. ZEILBERGER: Bijecting Euler's partitions-recurrence, Amer. Math. Monthly 92 (1985), 54-55.

2 Leonhard Euler: On the remarkable properties of the pentagonal numbers, Trans Jordan Bell, https://scholarlycommons.pacific.edu/euler-works/542