The Slope Problem Proof: a proof from the BOOK (by Martin Aigner and Gunter M Ziegler)

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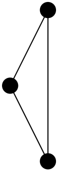
The following was conjectured by P.R.Scott in 1970, and finally proved by Peter Ungar in 1982.

Theorem

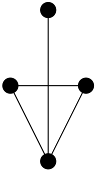
If $n \geq 3$ points in the plane do not lie on a single line, then they determined at least $n-1$ different slopes, where equality is possible only if n is odd and $n \geq 5$.

It often seems the the case with problems like this is the result seems very likely, but actually proving it is far from easy. For this result it is easy to come up with examples of the minimum possible. Some of the possible configurations are shown on the next page. There are at least four infinite families known, together with a further 102 isolated examples.



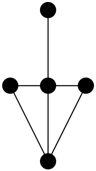
n=3

3 slopes



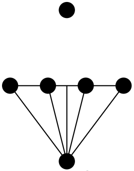
n=4

4 slopes



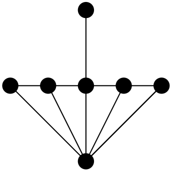
n=5

4 slopes



n=6

6 slopes



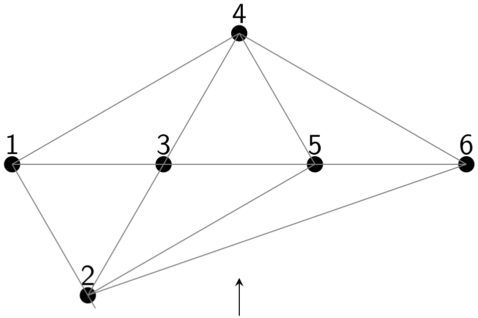
n=7

6 slopes

Proof of the theorem

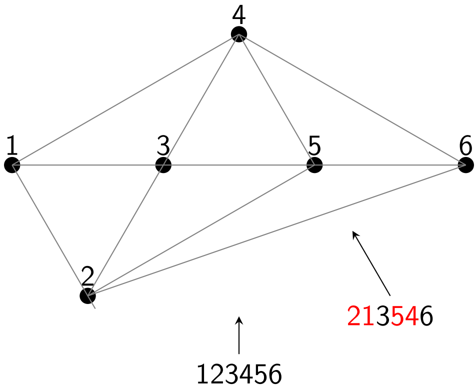
First of all, it suffices for us to prove that any even number $n = 2m$ of points not all co-linear determines at least n distinct slopes, since if n is odd, we can choose a subset of $n - 1$ points which are not all co-linear, and so since $n - 1$ would be even they must determine at least $n - 1$ distinct slopes.

If we view the points and all the possible from any given direction within the plane, then the positions of the points define a permutation of $1, 2, \ldots n$.

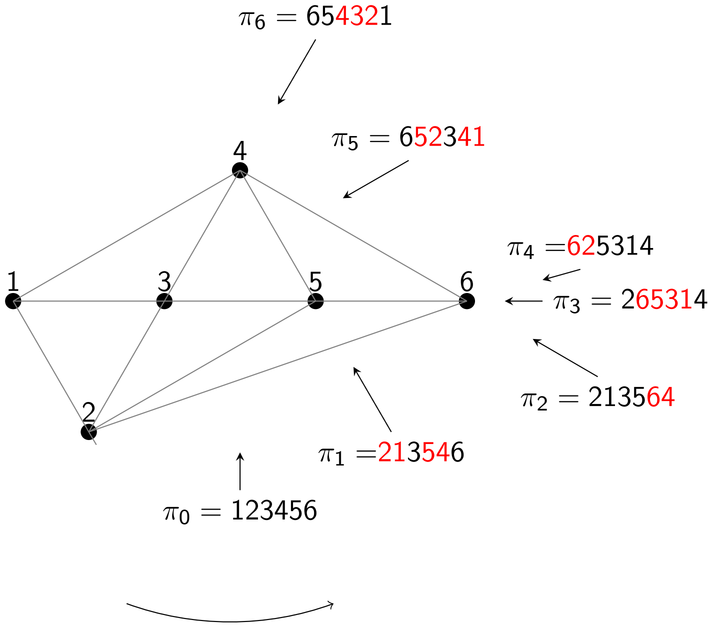


$\pi_0 = 123456$

As the direction of view is moved past a slope direction, the order of points on any line with that slope is reversed to give a new permutation.



When the view moves through 180 deg, the permutation completely reverses.



$\pi_6 = 654321$
$\pi_5 = 652341$
4
$\pi_4 = 625314$
1
3
5
6
$\pi_3 = 265314$
$\pi_2 = 213564$
2
$\pi_1 = 213546$
$\pi_0 = 123456$

So if there are t slopes and we continue to move the point of view indefinitely we have a sequence

$$\pi_0 \to \pi_1 \ldots \to \pi_t \to \pi_{t+1} \ldots \to \pi_{2t} \to \ldots$$

of permutations which is periodic with period $2t$, where the permutations completely reverse with period t.
If a line has an even number of points, pairs of points exchange places. If it has an odd number of points, then the middle one stays put in the permutation.

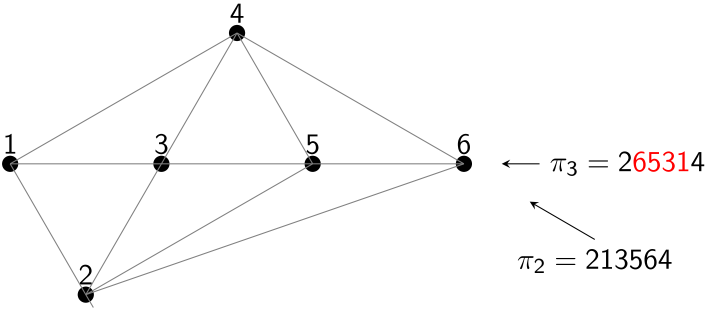
Since every pair of points is on exactly one line, the order of each pair in the sequence is switched exactly once every t moves. So between π_0 and π_t, only *increasing* sequences are reversed, and consequently this holds for all *even* half-periods, and for *odd* half-periods only *decreasing* sequences are reversed.

It is helpful to think of a division between the left and right-had sides of the permutation as a sort of barrier. Each point is moved from one side of the barrier to the other at least once every t moves.

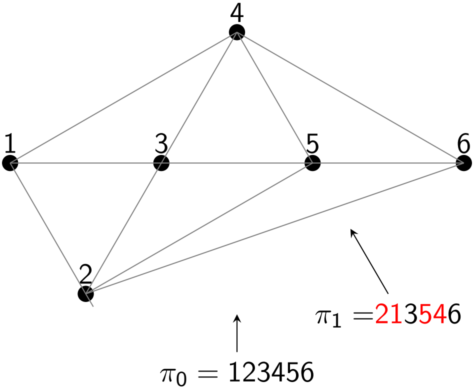
We identify three types of move of point of view according to the change in the permutation:

1. A **Crossing move of order** d is one in which d points cross the barrier in each direction.
2. A **Touching move** is the one where a sequence that borders barrier is reversed. Geometrically, this corresponds to a line that has exactly m points to one side of it.
3. An **Ordinary mode** is none of the above.

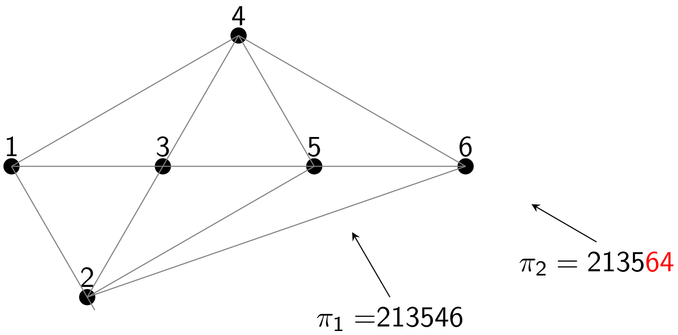
$\pi_2 \to \pi_3$ is a **crossing move** of order 2.



$\pi_0 \to \pi_1$ is a **touching move**.



$\pi_1 \to \pi_2$ is an **ordinary move**.



Any move reverses one or more contiguous sub-sequences corresponding to individual lines, leaving points on either side unchanged.
A crossing move of order d moves d points each way across the half-way barrier. Since all n points are moved across the barrier at least once every t moves, we can say:

$$\sum_{\substack{\text{all}\\\text{crossing}\\\text{moves}}} 2d \geq n$$

A crossing move changes an increasing (resp. decreasing) sub-sequence to an decreasing (resp. increasing) sub-sequence. Before another crossing move can occur, another increasing (resp decreasing) sub-sequence must be moved into the space spanning the barrier.
Because two points define a unique line, points from the subsequence to be moved have to be moved from the edges inwards, no more than one on each side at a time.

So to move that subsequence out of the barrier region to make room for a new increasing (resp. decreasing) subsequence, there must be at least $d-1$ ordinary moves , followed by a touching move.

$$\ldots\ldots C \to O_1 \to O_2 \to \ldots \to O_{d-1} \to T \to \ldots$$

Equally, before the crossing move of order d we are considering, there must be at least a touching move followed by at least $d-1$ ordinary moves.

$$\ldots \to T \to O_1 \to O_2 \to \ldots \to O_{d-1} \to \ldots C \to \ldots$$

So each crossing move is bracketed by a touching move and at least $2(d-1)$ ordinary moves. Which, including the crossing move itself makes $2d$ moves. So, considering all the crossing moves in each period of t moves we have:

$$t \geq \sum_{\substack{\text{all}\\\text{crossing}\\\text{moves}}} 2d \geq n$$

Q.E.D.

Generalisation

Janos Pach, Rom Pinchasi and Micha Sharir (2007) proved that:

Theorem

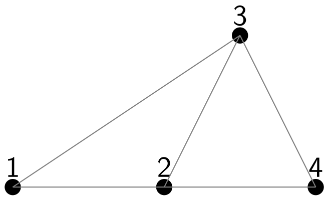
Let P be a set of points in R^3, not all of which are in a plane and no three on a line. We partially answer a question of Scott (1970) by showing that the connecting lines of P assume at least $2n - 3$ different directions if n is even and t least $2n - 3$ if n is odd. These bounds are sharp.

References

1 J.E. Goodman & R. Pollack: Combinatorial aspects on some problems in geometry, Congressus Numerantium **(32)** (1981) 383-394.

2 P.R.Scott: On the sets of directions determined by n points, Amer. Math. Monthly **(77)** (1970) 502-505.

3 P. Ungar: $2N$ non-collinear points determine at least $2N$ directions, J Combinatorial Theory, Ser. A **(33)** (1982) 343-347.



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