The Probabilistic Method

(mainly obtained from the book of the same name by Alon & Spencer.)
Ramsey numbers

In terms of graph colourings, the Ramsey number \( R(k, l) \) is the smallest number \( n \) such that any 2-colouring of \( K_n \) with \( k \) must contain either a monochromatic \( K_k \) or a monochromatic \( K_l \).

\[ R(3, 3) \leq 6 \]

**Proof**

Choose any vertex \( v \). Of the 5 vertex edges incident with \( v \) at least 3 will be the same colour. Say 3 of item \( (v,u) \), \( (v,w) \), \( (v,z) \) are blue. Then consider the edges \( (u,w) \), \( (u,z) \), \( (w,z) \). If one of them is blue, then there is a blue triangle. If all are red, there is a red triangle.

\[ R(3, 3) > 5 \text{ because:} \]

so \( R(3, 3) = 6 \)
Theorem \( R(h, k) > \left\lfloor 2^{\frac{k}{2}} \right\rfloor \) for all \( k \geq 3 \)

Proof

Consider a randomly 2-coloured \( K_n \)

For any \( k \)-subset \( A \) of the vertices, the probability that \( A \) be monochromatic is \( \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} \).

Therefore the probability that at least one such subset \( A \) is monochromatic is no more than \( \binom{n}{k} \cdot \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} \).

So if \( \binom{n}{k} \cdot \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} < 1 \), there must be a colouring of \( K_n \) with no monochromatic \( K_k \).

So to ensure a monochromatic \( K_k \) in every colouring

\[
\binom{n}{k} \cdot \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} \cdot \frac{n!}{k!(n-k)!} \cdot \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} < \frac{n^k}{k!} \cdot \frac{2}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}
\]

\[
= \frac{n^k}{k!} \cdot \left( \frac{n}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} \right)^k \cdot \frac{2}{k!} < 1 \quad \text{if} \quad n = \left\lfloor 2^{\frac{k}{2}} \right\rfloor
\]
So if \( \binom{n}{k} \frac{2}{2^{(\frac{n}{2})}} < 1 \)

there is a chance that at

there are colourings where there

is no monochromatic \( K_k \)

\[
\binom{n}{k} \frac{2}{2^{(\frac{n}{2})}} = \frac{n!}{k!(n-k)!} \frac{2}{2^{(\frac{n}{2})}}
\]

\[
< \frac{n^k}{k^k} \frac{2}{2^{\frac{n^k}{2}}} \leq 1 \quad \text{if } n^k \leq 2^{\frac{n}{2}}
\]

\[
\frac{n^k}{k^k} \frac{2}{2^{\frac{n^k}{2}}} \leq 1 \quad \text{if } n = 3
\]

\[
\frac{2}{2^{(\frac{n}{2})}} = \left\lfloor \frac{\sqrt{2}}{2} \right\rfloor - 2
\]

\[
\frac{2}{2^{(\frac{n}{2})}} = 2 \quad \text{for } n = 3
\]

\[
6 = R(3,3)
\]
Intersecting sets (Extremal Sets)

A family of sets \( F \) is intersecting if \( A \cap B \neq \emptyset \) for all \( A, B \in F \)

Sets \( m [n] = \{0, \ldots, n-1\} \)

EKR Theorem.

Any family \( F \) of \( k \)-subsets from \( n \) elements must have \( |F| \leq \binom{n-1}{k-1} \)

Lemma Any intersecting family of \( k \)-sets in \( m [n] \) cannot contain more than \( k \) intervals of length \( k \)

Proof

\[
\begin{align*}
&k \{ 0, 1, 2, \ldots, k-1 \\
&k \{ 1, 2, \ldots, k, k+1 \\
&k \{ 2, \ldots, k \quad k+1 \quad k-1, k, \ldots, 2k-2 \\
\end{align*}
\]
Katona's proof (1972)

Take all the members of \( F \) and write down all possible ways I can label the elements of each subset with 0, ..., \( k \). Each chosen subset is labelled 0, 1, ..., \( k \).

Each member of \( F \) can be ordered 0, ..., \( k-1 \) in \( k! \) ways and the rest of \( [n] \) can be labelled in \( (n-k)! \) ways.

So that makes a total of \( |F| \frac{k!}{k!} (n-k)! \)

Now if we consider the whole set and consider the circular permutations. There are \( (n-1)! \) of them. And each of them contains \( k \) intersecting intervals.

But each of the labellings we considered is for \( F \) must be one of those labellings.

So \( |F| \frac{k!}{k!} (n-k)! \leq k(n-1)! \)
\[
|F| \leq \frac{k \cdot (n-1)!}{k! \cdot (n-k)!} = \frac{(n-1)!}{(k-1)! \cdot (n-k)!} = \binom{n-1}{k-1}
\]
Some Results using the Probabilistic Method

Alon & Spencer 'The Probabilistic Method'

"Sum-free sets"
A set of two integers $A$ where there is no $x, y, z \in A$ such that $x + y = z$.
(you can $x = y$ etc)

(Erdős 1965) If $X$ is a set of non-zero two integers of size $n$
Then $X$ contains a sum-free subset of size at least $n/3$.

Proof let $p$ be a prime $> \max (X)$
and let $p = 3k + 1$ (som for convenience)
Let $K = \{ k, k+1, \ldots, 2k-1 \}$
Choose $a \in \{ 1, \ldots, p-1 \}$ randomly
Then for any $x \in X$ the probability that
$$\text{an } (\text{mod } p)$$
$$\text{Prob}(ax \text{ mod } p \in K) = \frac{|K|}{p-1} = \frac{k}{3k} = \frac{1}{3}$$

So summing over all of $X$
$$E(|aX \cap K|) = \frac{n}{3}$$

So as $a$ ranges over all all values
either $|aX \cap K|$ are all the same or
there are some that are larger and some that are
than the expectation.
So \( a \in \{1, 2, \ldots, p-1\} \)

such that \(|aXaK| \geq \frac{n}{3}\)

\( \exists b \) such that \( ab \equiv 1 \mod p \)

\( |XaKb| \geq \frac{n}{3} \)

So \( X \) contains a sum-free subset

of size at least \( \frac{n}{3} \)

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The easiest examples of sum-free subsets tend to be of size \( \sqrt{n} \) or \( \frac{n+1}{2} \).

E.g. \( \{k, k+1, \ldots, 2k-1\} \) in \( \{1, 2, \ldots, 2k-1\} \)

and all odd numbers in \( \{1, 2, \ldots, 2k-1\} \),

but there are examples in the literature of subsets that approach Erdos' lower bound:

\( \{2, 3, 4, 5, 6, 8, 10\} \) (Klarner) \( 3/7 = 0.43 \)

\( \{1, 2, 3, 4, 5, 6, 8, 9, 10, 18\} \) (Melvyn) \( 2/5 = 0.4 \)

Lewko 2010:

\( \{1, 2, 3, \ldots, 18, 20, 22, 24, 25, 26, 27, 30, 34, 50, 54\} \)

\( 11/28 = 0.393 \)

Finally, Eberhard, Green, Manners (2014) vindicated Erdos' lower bound with the result:

"For every \( \epsilon > 0 \) \( \exists \) a set of \( n \) integers \( A \) such that

for all \( A' \subseteq A \) where \( |A'| \geq \left( \frac{1}{3} + \epsilon \right) n \)

\( A' \) contains distinct \( x, y, 3 \) such that \( x + y = 3 \)."
An independent set of vertices is a set $S$ of vertices with no connecting edges.

Graph $G = (V, E)$

Let $d_v$ be the degree of vertex $v$ in $G$. Let $\alpha(G)$ be the size of the largest independent set in $G$.

(Caro & Wei)

**Theorem**

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{1 + d_v}$$

Choose a random ordering of the vertices $V$.

Define the set $I = \{ v \in V : (v, w) \in E \text{ then } v < w \}$

Let $X_v = 1$ if $v \in I$

$= 0$ otherwise.

Then $\text{prob}(X_v = 1) = \frac{1}{1 + d_v}$

Let $X = \sum_{v \in V} X_v$

$$E(X) = \sum_{v \in V} \frac{1}{1 + d_v}$$

As we run through all orderings of $V$ either all $S$, $I$ must be of some size or there is at least one that is bigger than $E(X)$.

So if $I$ s.t. $|I| \geq \sum_{v \in V} \frac{1}{1 + d_v}$

But if $x, y \in I$ $(x, y) \notin E$ because otherwise $x < y$ and $y < x$, so $I$ is independent and so $\alpha(G) \geq \sum_{v \in V} \frac{1}{1 + d_v}$