

## The Probabilistic Method

(mainly obtained from the book of  
the same name by Alon & Spencer.)

# Ramsey numbers

In terms of graph colourings, the Ramsey number  $R(k, l)$  is the smallest number  $n$  such that any 2-colouring of  $K_n$  ~~with~~ must contain either a monochromatic  $K_k$  or a monochromatic  $K_l$ .

$R(3, 3) \leq 6$  Proof Colour a  $K_6$  red and blue.

Choose any vertex  $v$ . Of the 5 ~~vertices~~ edges incident with  $v$  at least 3 will be the same colour. Say 3 of them  $(v, u), (v, w), (v, z)$  are blue. Then consider the edges  $(u, w), (u, z), (w, z)$ . If one of them is blue, then there is a blue triangle. If all are red, there is a red triangle.

$R(3, 3) > 5$  because:



so  $R(3, 3) = 6$

Erdos 1947

Theorem  $R(n, k) > \lfloor 2^{\lfloor n/2 \rfloor} \rfloor$  for all  $k \geq 3$

Proof

Consider a randomly 2-coloured  $K_n$

For any  $k$ -subset  $A$  of the vertices, the probability that  $A$  be monochromatic is  $\frac{2}{2^{\binom{k}{2}}}$ .

Therefore the probability that at least one such subset is monochromatic is no more than  $\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}}$

So if  $\binom{n}{k} \frac{2}{2^{\binom{k}{2}}} < 1$ , there must be a colouring of  $K_n$  with no monochromatic  $K_k$

~~So to ensure a monochromatic  $K_k$  in every colouring~~

$$\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} \cdot \frac{n!}{k!(n-k)!} \cdot \frac{2}{2^{\binom{n-k}{2}}} < \frac{n^k}{k!} \cdot \frac{2}{2^{\lfloor \frac{k^2}{2} - \frac{k}{2} \rfloor}}$$
$$= \frac{n^k}{k!} \cdot \left( \frac{n}{2^{\lfloor \frac{k^2}{2} - \frac{k}{2} \rfloor}} \right)^k \cdot \frac{2}{k!} < 1 \text{ if } n = \lfloor 2^{\lfloor \frac{k^2}{2} \rfloor} \rfloor$$

$$\text{So if } \binom{n}{k} \frac{2}{2^{\binom{n}{2}}} < 1$$

there is a chance that at  
there are colourings where there  
is no monochromatic  $K_k$

$$\binom{n}{k} \frac{2}{2^{\binom{n}{2}}} = \frac{n!}{k!(n-k)!} \frac{2}{2^{\binom{n}{2}}}$$

$$< \frac{n^k}{k!} \frac{2}{2^{n/2 - k/2}} < 1 \text{ if } \left\lceil 2^{n/2} \right\rceil$$

$$\binom{n}{k} \frac{2}{2^{\binom{n}{2}}}$$

~~$$\frac{n^k}{k!} \frac{2}{2^{n/2 - k/2}} < 1 \text{ if } \left\lceil 2^{n/2} \right\rceil$$~~

$$= \left\lceil 2^{n/2} \right\rceil - 2$$

$$\text{if } n = 3$$

$$\binom{2}{2} \frac{2}{2^{\binom{2}{2}}} = 2 < 6 = R(3,3)$$

# Intersecting sets (External sets)

A family of ~~sets~~ sets  $\mathcal{F}$  is intersecting if  $A \cap B \neq \emptyset$  for all  $A, B \in \mathcal{F}$

---

Sets on  $[n] = \{0, \dots, n-1\}$

## EKR Theorem.

Any <sup>intersecting</sup> family  $\mathcal{F}$  of  $k$ -sets on  $[n]$  must have  $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Lemma Any intersecting family of  $k$ -sets on  $[n]$  cannot contain more than  $k$  intervals of length  $k$

Proof

k	{	0, 1, 2, ..., k-1
		1, 2, ..., k, k+1
		2, ..., k, k+1
		k-1, k, ..., 2k-2

## Katona's proof (1972)

Take all the members of  $\mathcal{F}$  and write down all possible ways I can label ~~the elements of each subset~~ with  $0, \dots, k-1$   $[n]$  so that a chosen subset is labelled  $0, 1, \dots, k-1$

~~Each~~ Each member of  $\mathcal{F}$  can be ordered  $0, \dots, k-1$  in  $k!$  ways and the rest of  $[n]$  can be labelled in  $(n-k)!$  ways

So that makes  $\mathcal{F}$  a total of

$$|\mathcal{F}| k! (n-k)!$$

Now if we consider the whole set and consider the circular permutations. There are  $(n-1)!$  of them. And each of them contains  $k$  intersecting intervals

But each of the labellings we considered ~~is~~ for  $\mathcal{F}$  must be one those labellings

$$\text{So } |\mathcal{F}| k! (n-k)! \leq k(n-1)!$$

$$|F| \leq \frac{k}{k!} \cdot \frac{(n-1)!}{(n-k)!}$$

$$= \frac{(n-1)!}{(k-1)! (n-k)!} = \binom{n-1}{k-1}$$

## Some Results using the Probabilistic Method

Alon & Spencer 'The Probabilistic Method.'

"Sum-free sets"

A set of +ve integers  $A$  where there is no  $x, y, z \in A$  such that  $x + y = z$ .  
(you can  $x = y$  etc)

---

(Erdos 1965) If  $X$  is a set of non-zero +ve integers of size  $n$   
Then  $X$  contains a sum-free subset of size at least  $n/3$ .

Proof Let  $p$  be a prime  $> \max(X)$   
and let  $p = 3k + 1$  (con for convenience)

Let  $K = \{k, k+1, \dots, 2k-1\}$

Choose  $a \in \{1, \dots, p-1\}$  randomly

~~So~~ For any  $x \in X$  the probability that  $ax \pmod{p}$

$$\text{Prob}(ax \pmod{p} \in K) = \frac{|K|}{p-1} = \frac{k}{3k} = \frac{1}{3}$$

So summing over all of  $X$

$$E(|aX \cap K|) = \frac{n}{3}$$

So as  $a$  ranges over all all values

either  $|aX \cap K|$  are all the same or there are some that are larger and some that are than the expectation.

So  $\exists$  an  $a \in \{1, 2, \dots, p-1\}$

such that  $|aX \cap K| \geq \frac{n}{3}$

$\exists b$  such that  $ab \equiv 1 \pmod{p}$

$\therefore |X \cap bK| \geq \frac{n}{3}$

So  $X$  contains a sum-free subset  
of size at least  $\frac{n}{3}$

---

↳ The easiest examples of sum-free <sup>sub-</sup>sets tend  
to be of size  $n/2$  or  $(n+1)/2$

eg  $\{k, k+1, \dots, 2k-1\}$  in  $\{1, 2, \dots, 2k-1\}$

and all odd numbers in  $\{1, 2, \dots, 2k-1\}$ ,

but there are examples in the literature of subsets  
that approach Erdős' lower bound:

$\{2, 3, 4, 5, 6, 8, 10\}$  (Klarner)  $3/7 = 0.43$

$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 18\}$  (Meluf)  $2/5 = 0.4$

Lewko ~~2010~~ 2010:

$\{1, 2, 3, \dots, 18, 20, 22, 24, 25, 26, 27, 30, 34, 50, 54\}$   
 $n/2.8 = 0.393$

Finally Eberhard, Green, Parnowski (2014) vindicated  
Erdős' lower bound ~~with~~ with the result:

"For every  $\epsilon > 0 \exists$  a set of  $n$  integers  $A$  such that  
for all  $A' \subset A$  where  $|A'| \geq (\frac{1}{2} + \epsilon)n$   
 $A'$  contains distinct  $x, y, z$  such that  $x + y = z$ ."

An independent set of vertices is a set of vertices with no connecting edges

$$\text{Graph } G = (V, E)$$

Let  $d_v$  = degree of vertex  $v$  in  $G$ .  
 $\alpha(G)$  = size of largest independent set in  $G$ .  
 (Caro & Wei)

Theorem

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{1+d_v}$$

Choose a random ordering of the vertices

Define the set  $I = \{v \in V : (v, w) \in E \text{ then } v < w\}$

Let  $X_v = 1$  if  $v \in I$   
 $= 0$  otherwise

Then  $\text{prob}(X_v = 1) = \frac{1}{1+d_v}$

= prob that  $v$  comes 1st in ordering of  $v$  and its neighbours

Let  $X = \sum_{v \in V} X_v$

$$E(X) = \sum_{v \in V} \frac{1}{1+d_v}$$

As we run through all orderings of  $V$  either all  $I$  must be of some size or there is at least one that is bigger than

$E(X)$

so  $\exists I$  st  $|I| \geq \sum_{v \in V} \frac{1}{1+d_v}$

But if  $x, y \in I$   $(x, y) \notin E$  because otherwise  $x < y$  and  $y < x$ .  
 so  $I$  is independent and so  $\alpha(G) \geq \sum_{v \in V} \frac{1}{1+d_v}$