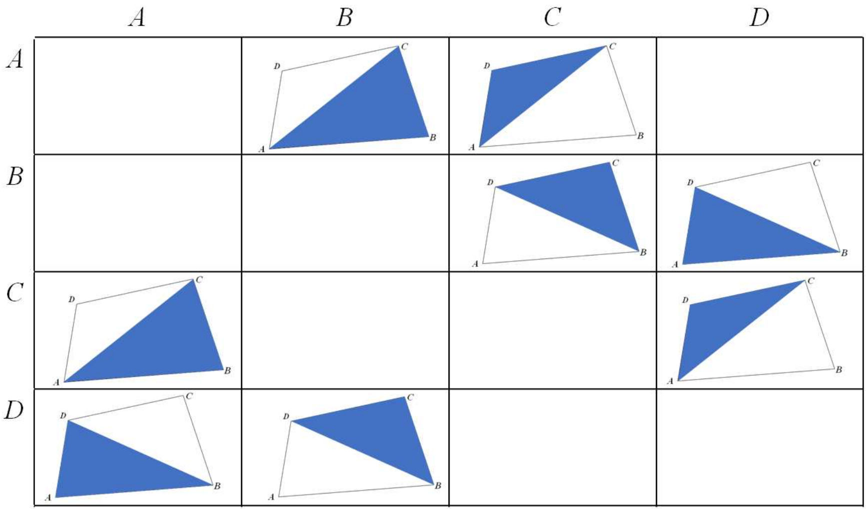
A Matrix of Triangle Areas



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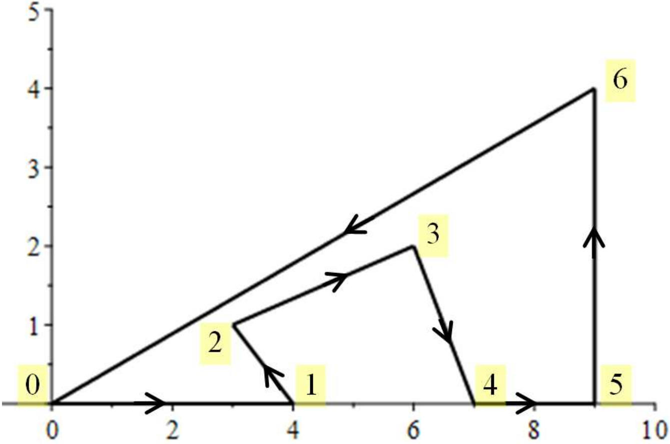
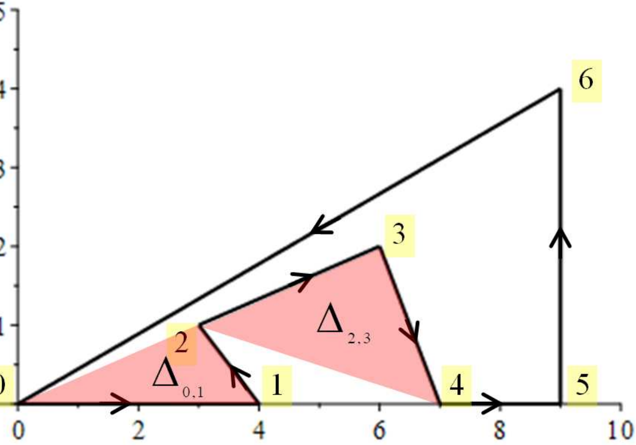
Triangles in polygons

Let P be a simple polygon on n vertices, $0, 1, \ldots, n-1$, oriented counterclockwise. We are interested in the areas of the triangles formed by joining vertices of P to 'opposite edges'.



Signed areas

Denote by Δ_{ij} the area of the triangle on polygon vertices $i, j, j+1$, the numbering taken modulo n. This area is taken as positive or negative according to whether $i, j, j+1, i$ has counterclockwise or clockwise orientation relative to the orientation of the polygon.

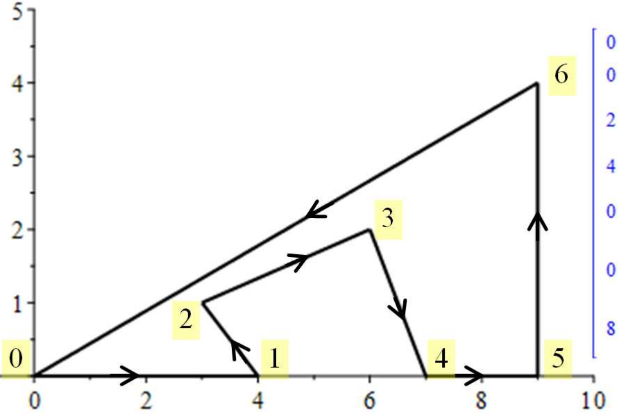


Left we have highlighted areas $\Delta_{0,1} = 2$ and $\Delta_{2,3} = -7/2$.

Note also $\Delta_{0,2} = \Delta_{0,4} = 0$.

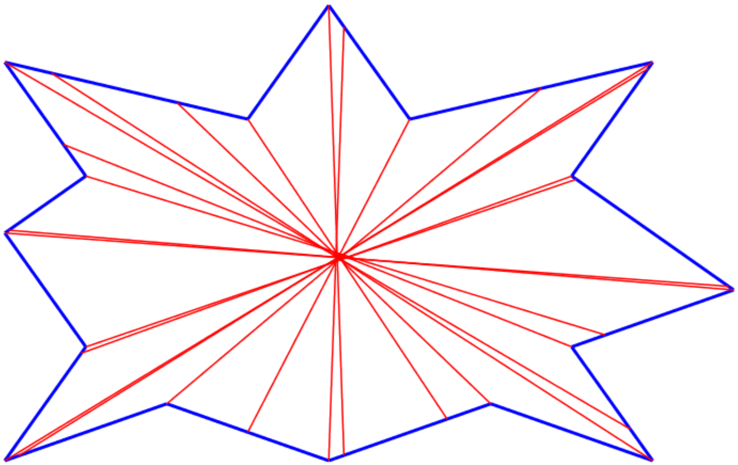
The Delta matrix

For an n-vertex polygon the Δ_{ij} form an $n \times n$ matrix. The main and 1st lower diagonals are zero. The 1st upper diagonal equals the 2nd lower diagonal.



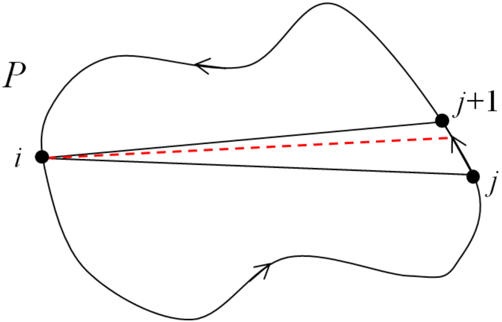
$$\begin{bmatrix} 0 & 2 & 0 & -7 & 0 & 18 & 0 \\ 0 & 0 & -2 & -3 & 0 & 10 & 8 \\ 2 & 0 & 0 & -\frac{7}{2} & 1 & 12 & \frac{3}{2} \\ 4 & -2 & 0 & 0 & 2 & 6 & 3 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & 0 & 0 & 4 & 14 \\ 0 & -\frac{5}{2} & -\frac{9}{2} & 2 & 0 & 0 & 18 \\ 8 & -\frac{9}{2} & \frac{3}{2} & 4 & 4 & 0 & 0 \end{bmatrix}$$

Application 1: Finding bisecting chords in polygons



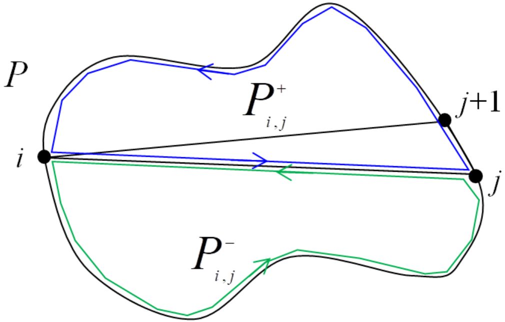
Looking for bisecting chords

We are interested in the $\Delta_{i,j}$, firstly because they help locate bisecting chords in polygon P. Specifically, does the collection of chords subtending edge $[j, j+1]$ from vertex i, denoted $\langle i, j \rangle$, contain one which bisects the area of P?



Tracing sub-polygons

Trace $P^+_{i,j}$ by following chord $[i,j]$ and then edges of P counterclockwise to i. Denote its area by $A^+_{i,j}$.
Trace $P^-_{i,j}$ by following chord $[j,i]$ and then edges of P counterclockwise to to j. Denote its area by $A^-_{i,j}$.



Calculate $r_{i,j} = \left(A^+_{i,j} - A^-_{i,j}\right)/2\Delta_{i,j}$. A value in $[0,1]$ locates a bisection point on edge $[i,j]$.

Looking for 'bisecting' $r_{i,j}$ values

Our strategy is to take each polygon vertex in turn and test each chord from that vertex in turn to identify which is bisecting. Luckily we can calculate the r_{ij} values without ever having to construct any of the auxiliary polynomials P^+ and P^-:

Proposition

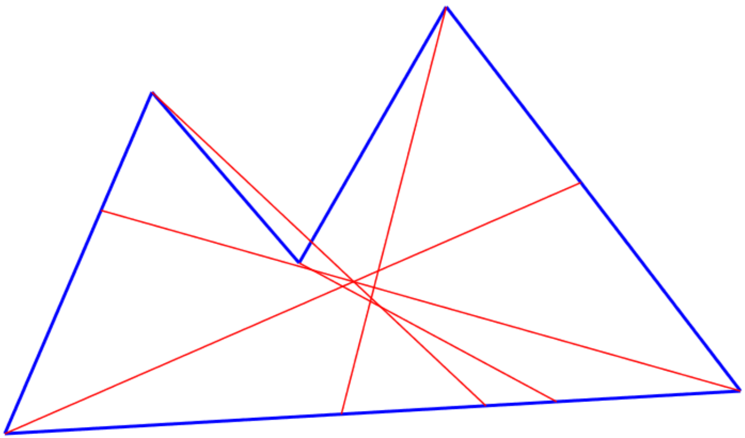
1. Let P have area A_P. For $i = 0, \ldots, n-1$ and $j = i+1, \ldots, i+n-2$,

$$r_{i,j} = \begin{cases} A_P/2\Delta_{i,j} & \text{if } j = i+1 \text{ and } \Delta_{i,j} \neq 0 \\ (r_{i,j-1} - 1)\Delta_{i,j-1}/\Delta_{i,j} & \text{if } j \geq i+2 \text{ and both } \Delta_{i,j} \text{ and } \\ & \Delta_{i,j-1} \text{ are nonzero.} \end{cases}$$

2. Suppose that $\Delta_{i,j} = 0$ but that $\Delta_{i,j-1}$ and $\Delta_{i,j+1}$ are both nonzero. Then

$$r_{i,j+1} = (r_{i,j-1} - 1)\Delta_{i,j-1}/\Delta_{i,j+1}.$$

Application 2: Testing for bisection convexity



Bisection convexity

Definition A polygon is called **bisection-convex** if any straight line which bisects its area contains either exactly two points on the boundary of the polygon, or contains one point on the boundary and one edge of the polygon. If the latter case does not occur the polygon is called strictly bisection-convex.

(Fechtor-Pradines, N., “Bisection envelopes”, *Involve*, Vol. 8, No. 2, 2015.)

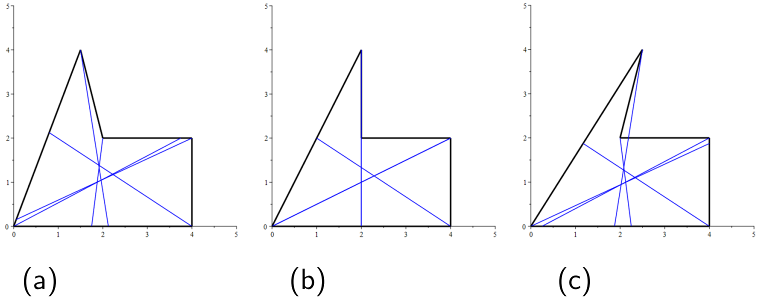
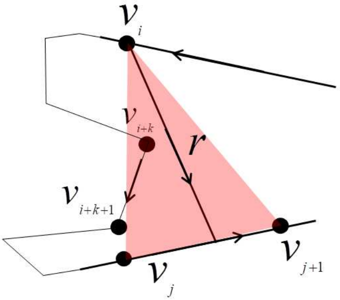


Figure : (a) Strictly bisection-convex; (b) non-strictly bisection-convex; (c) not bisection-convex

Testing for bisection convexity

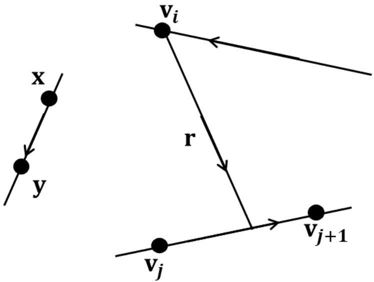
Proposition A polygon P is bisection convex if and only if no bisecting chord originating at a vertex of P has a point exterior to P.

Means we just have to check each vertex-opposite edge bisecting chord to see if it intersects P elsewhere.



Testing for bisection convexity using $\Delta_{i,j}$s

The text-book check for line intersection is invoked below: the vertices are labelled with position vectors while $\mathbf{r}$ is the direction vector corresponding to the bisecting chord at $\mathbf{v}_i$. The polygon edge joining vertices $\mathbf{x}$ and $\mathbf{y}$ will intersect with the straight line passing through $\mathbf{v}_i$ in direction $\mathbf{r}$ if and only if $\mathbf{x}$ and $\mathbf{y}$ lie in different half-planes in relation to this straight line. This occurs if and only if the two cross products $(-\mathbf{v}_i + \mathbf{x}) \times \mathbf{r}$ and $(-\mathbf{v}_i + \mathbf{y}) \times \mathbf{r}$ have different signs. So the strategy is to calculate the series of cross products $(-\mathbf{v}_i + \mathbf{v}_{i+k}) \times \mathbf{r}$, $k = 1, \ldots, n-1$ and look for sign changes.



Of course these are cross products in the plane so they are effectively scalars, $(a, b) \times (c, d) = ad - bc$.

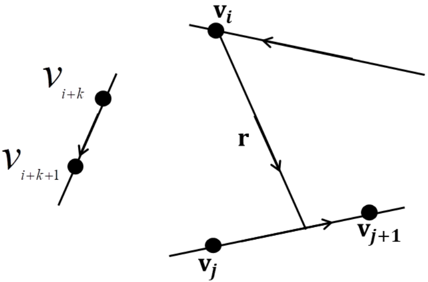
Vector intersections via Δ_{ij}s

Proposition Define the sequence F_i, $i \geq 0$, by

$$F_0 = 0 \text{ and, for } k \geq 1, F_k = F_{k-1} + \Delta_{j,i+k-1} - \Delta_{i,i+k-1}.$$

Then for $k = 1, \ldots, n-1$,

$$(-\mathbf{v}_i + \mathbf{v}_{i+k}) \times \mathbf{r} = 2r_{i,j}\left(\Delta_{i,j} - \Delta_{i+k,j}\right) + 2F_k.$$

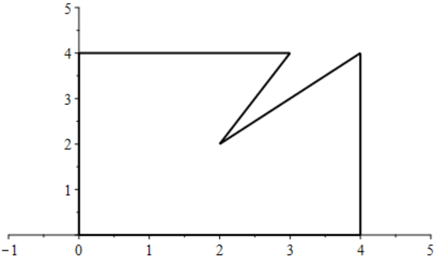


Properties of the Delta matrix 1: Rank

0	−2	2	−4	8	3/2	12	9/2	13/2	8	0	21/2	−7/2	2	−2	0
0	0	2	−2	11/2	3/2	9	3	6	11/2	3/2	8	−1	5/2	1/2	3/2
−2	0	0	−2	4	−1/2	8	1/2	17/2	4	4	17/2	1/2	4	2	4
−2	2	0	0	3/2	−1/2	5	−1	8	3/2	11/2	6	3	9/2	9/2	11/2
−4	2	−2	0	0	−5/2	4	−7/2	21/2	0	8	13/2	9/2	6	6	8
−3/2	7/2	1/2	3/2	0	0	3	−3/2	7	0	6	4	9/2	9/2	6	6
−3/2	11/2	1/2	7/2	−5/2	0	0	−3	13/2	−5/2	15/2	3/2	7	5	17/2	15/2
3/2	13/2	7/2	9/2	−3/2	3	0	0	5/2	−3/2	9/2	−1/2	6	3	15/2	9/2
3	9	5	7	−7/2	9/2	−3	0	0	−7/2	9/2	−4	8	5/2	19/2	9/2
7/2	13/2	11/2	9/2	0	5	1	5/2	0	0	2	−1	9/2	3/2	6	2

Easy properties of $\Delta(P)$ the matrix of $\Delta_{i,j}$s

1. The diagonal and 1st lower diagonal entries are zero.
2. The first upper diagonal is identical to the second lower diagonal, the same triangle areas being subtended from opposite directions. That is, for $i = 0, 1, \ldots, n-1$, $\Delta_{i,i+1} = \Delta_{i+2,i}$.
3. In row i the first and second upper diagonal entries sum to same as the first and second lower diagonal entries in row $i+3$. That is $\Delta_{i,i+1} + \Delta_{i,i+2} = \Delta_{i+3,i} + \Delta_{i+3,i+1}$.



$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

Linear combinations of rows of $\Delta(P)$

A 3×3 diagonal block of $\Delta(P)$ has the following form

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ x & 0 & 0 \end{bmatrix}$$

The Easy Properties extend this to the row immediately beneath:

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ x & 0 & 0 \\ x+y-z & z & 0 \end{bmatrix}$$

Suppose $x \neq 0$. Write $\alpha_1 = z/x, \alpha_2 = -y/x, \alpha_3 = 1 - \alpha_1 - \alpha_2$. Then this extension is the linear combination

$$\text{Row } 4 = \alpha_1 \times \text{Row } 1 + \alpha_2 \times \text{Row } 2 + \alpha_3 \times \text{Row } 3$$

Surprisingly (to me) this same linear combination extends to the remainder of this same row.

Less immediate property of $\Delta(P)$

The matrix $\Delta(P)$ is determined by its first three rows. Specifically, $\Delta(P)$ has rank 3.

$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

By the easy properties, a diagonal 3×3 block determines the 3 entries immediately below.

As we have seen, we can write these 3 entries explicitly as a linear combination of the rows above.

And this same linear combination applies across the whole row.

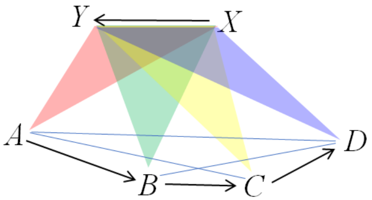
$$\text{Row } 4 \quad = \quad \frac{z}{x} \times \text{Row } 1 - \frac{y}{x} \times \text{Row } 2 + \frac{x+y-z}{x} \times \text{Row } 3,$$

provided $x \neq 0$.

A theorem about triangles

Suppose we have 6 points, A, B, C, D, X, Y, arranged in counterclockwise order in the plane. Using $|ABC|$, etc, to denote the area of the triangle on the points indicated, we have

$$|ABC| \times |XYD| + |ACD| \times |XYB| = |BCD| \times |XYA| + |ABD| \times |XYC|.$$



In the special case where the four triangles on base XY all had unit area this would just equate two ways of expressing the quadrilateral area $|ABCD|$. In fact it remains true in general.

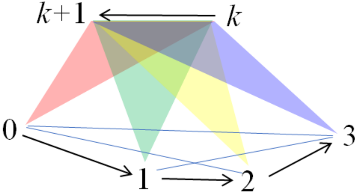
Applying the triangles theorem to the $\Delta_{i,j}$s

Without loss of generality take A, B, C, D to be the first four vertices, $0, 1, 2, 3$, of polygon P. The triangle theorem becomes

$$\Delta_{0,1} \times |XY3| + \Delta_{0,2} \times |XY1| = \Delta_{1,2} \times |XY0| + \Delta_{3,0} \times |XY2|.$$

Further, taking X, Y to be a subsequent edge $[k, k+1]$ of P, this becomes

$$\Delta_{0,1} \times \Delta_{3,k} + \Delta_{0,2} \times \Delta_{1,k} = \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$



0	$\Delta_{0,1}$	$\Delta_{0,2}$	$\cdots$	$\Delta_{0,k}$	$\cdots$
0	0	$\Delta_{1,2}$	$\cdots$	$\Delta_{1,k}$	$\cdots$
$\Delta_{0,1}$	0	0	$\cdots$	$\Delta_{2,k}$	$\cdots$
$\Delta_{0,1} + \Delta_{0,2} - \Delta_{1,2}$	$\Delta_{1,2}$	0	$\cdots$	$\Delta_{3,k}$	$\cdots$

Rearranging, provided that $\Delta_{0,1} \neq 0$,

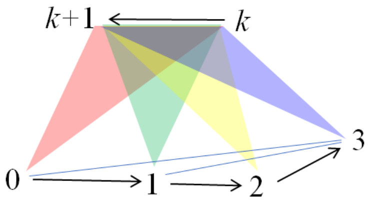
$$\Delta_{3,k} = \alpha_1 \Delta_{0,k} + \alpha_2 \Delta_{1,k} + \alpha_3 \Delta_{2,k},$$

with the α_i as on the earlier slide.

Dealing with zero-area $\Delta_{i,j}$s

If three consecutive polygon vertices are collinear then we have a zero area triangle.

E.g. here $\Delta_{0,1} = 0$ in the application of the triangle theorem, so we cannot write $\Delta_{3,k}$ in terms of the three previous rows of the matrix.

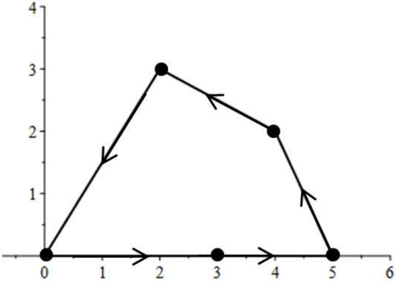


$$\Delta_{0,1} \times \Delta_{3,k} + \Delta_{0,2} \times \Delta_{1,k} = \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$

In fact, these three previous rows are found to be already linearly dependent:

$$0 = -\Delta_{0,2} \times \Delta_{1,k} + \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$

Zero-area $\Delta_{0,1}$: example



$$\begin{bmatrix} 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 2 & \frac{5}{2} & \frac{9}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{15}{2} \\ 3 & 2 & 0 & 0 & 4 \\ \frac{9}{2} & 3 & \frac{3}{2} & 0 & 0 \end{bmatrix}$$

$\Delta_{0,1} = 0$ and so

$\Delta_{1,2} \times \Delta_{0,k} - \Delta_{0,2} \times \Delta_{1,k} + \Delta_{3,0} \times \Delta_{2,k} = 0,$

or

$2 \times \Delta_{0,k} - 5 \times \Delta_{1,k} + 3 \times \Delta_{2,k} = 0.$

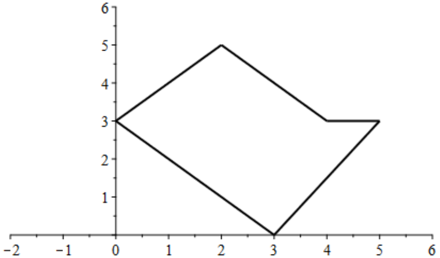
Thus, $k = 3$:

$$2 \times 4 - 5 \times \frac{5}{2} + 3 \times \frac{3}{2} = 0.$$

$k = 4$:

$$2 \times 0 - 5 \times \frac{9}{2} + 3 \times \frac{15}{2} = 0.$$

Properties of the Delta matrix 2: Eigenvectors



$$sm := \begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix}$$

$$csm := q^5 - 64\,q^3 - \frac{6279}{8}\,q^2$$

Eigenvalues and eigenvectors

A complex number λ is an eigenvalue of square matrix X if, for some complex row vector v,

$$\begin{aligned} vX &= \lambda v \qquad && v \text{ is a left eigenvector} \\ Xv^T &= \lambda v^T \qquad && v^T \text{ is a right eigenvector} \end{aligned}$$

E.g. $X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -2 & -2 & -2 \end{pmatrix}$

Eigenvalues are 0 and $1 \pm 2i$.

Right eigenvectors are

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1+i)/2 \\ (-2+i)/2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1-i)/2 \\ (-2-i)/2 \\ 1 \end{pmatrix}.$$

A right eigenvector of Δ

If all rows of a square matrix X have the same sum s then s is an eigenvalue of X and the all-ones column vector is a right eigenvector.

$$X \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

In the matrix $\Delta(P)$ for a polygon P, the i-th row sum is $\Delta_{i,i+1} + \ldots + \Delta_{i,i+n-2}$ (indexing modulo n). This sums a collection of triangles which partition the interior of P. Therefore all rows of $\Delta(P)$ sum to A_P, the area of the polygon.

(This is not quite obvious, unless P is convex, because some of the triangles may lie partially outside the boundary of P. That the overall sum is A_P is the content of the Shoelace Formula for polygon area.)

Thus A_p is an eigenvalue of $\Delta(P)$ with corresponding right eigenvector the all-ones vector.

Some left eigenvectors of Δ

Suppose rows $i, \dots i+k$ of a square matrix X are linearly dependent. This is the same as saying that summing these rows with the appropriate weights, say, $\alpha_1, \dots \alpha_k$, will give the zero vector.

Then multiplying X on the left by the row vector $v = (0, \dots, 0, \alpha_1, \dots, \alpha_k, 0, \dots, 0)$ will give a row of zeros. That is, $vX = 0 \times v$, so v is a left eigenvector corresponding to an eigenvalue of zero.

We have seen that $\Delta(P)$, for an n-vertex polygon P, has $n-3$ such dependencies. So zero is an eigenvalue of Δ of multiplicity $n-3$. We can construct the corresponding left eigenvectors as shown earlier, making $n-3$ applications of the triangle theorem.

The characteristic polynomial of Δ

The eigenvalues of an $n \times n$ matrix X are precisely the roots of the degree-n polynomial in λ defined as $c(X, \lambda) = \det(X - \lambda I)$. This is the **characteristic polynomial** of X.

From the previous slides we know we can almost write down $c(\Delta(P), \lambda)$ explicitly, factorised in terms of its roots:

$$c(\Delta(P), \lambda) = \lambda^{n-3}(\lambda - A_P)(\lambda^2 + a\lambda + b),$$

where A_P is the area of polygon P. The final factor is a quadratic which factorises as

$$(\lambda + A_P + Qi)(\lambda + A_P - Qi),$$

where Q is a, generally irrational, real number which will be

the subject of a future talk.