Playing Several Games at Once

A Look at Chapter 7 of On Numbers and Games John H Conway

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Nota bene

- Conway’s text is generally very concise and economical and my attempts to paraphrase it generally less so.
- Therefore, a lot of the text and diagrams on these slides is taken verbatim from Chapter 7 of On Numbers and Games by John H. Conway, second edition, A. K. Peters Ltd, 2001.
- Conway’s text is indicated by quotation marks.
- I have had to change the notation slightly as Conway uses a rotated equals sign (two short close vertical lines) to indicate the fuzzy relationship. I use * for this, which is also the value of the simplest fuzzy game.

A Class of Games

- Two Person games with players Left and Right
- "Our sympathies are with Left" – i.e. a good situation for Left is positive a good situation for Right is negative
- Within a game are positions *P*
- At each position *P* there are rules that restrict Left to certain moves called the left options of *P signified by* P^L and similarly Right is restricted to P^R the right options of *P*.
- "Any position P is completely determined by its Left and Right options, so we can write $P = \{P^L \mid P^R\}$

E.g.

- “If in a game there is a position *P*, from which
- “Left may move to any one of certain positions A, B or C (only)
- “And Right may move only to the position D
- “Then we write $P = \{A, B, C \mid D\}$

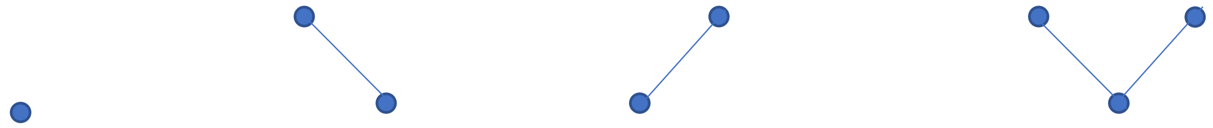
Game Over

- A game *ends* when the player who is called upon to move is unable do so
- E.g. given position {|U, V, W, X} with Left about to move the game ends
- A player who is unable to move when called upon to do so is the loser
- “This is a natural convention, for since we normally consider ourselves as losing when we cannot find any good move, we should obviously lose when we cannot find any move at all!
- All games end – there are no infinite sequences of positions each of which is an option of its predecessor (including when these alternate Left and Right).

Starting Position & Game ID

- “Each game *G* has its own proper starting position
- But from any position *P* there is a shorter game starting at *P*
- We can Identify this game with *P*
- And hence every game *G* is identified with its starting position
- “It follows from these conventions that a game can be represented by trees with:
  - The positions being represented by nodes (the initial position being the lowest node or *root*)
  - Legal moves represented by branches with Left’s move to the left and Right’s move to the right

Example Game Trees



0 = {|}

1 = {0|}
= {{|}|}

-1 = {|0}
= {|{|}}

• = {0|0}
= {{|}|{|}}

The simplest game of all is 0 the *endgame* whoever starts loses so there is a winning strategy for the second player

In 1 = {0|} there is a winning strategy for Left whether they go first or second

In -1 = {|0} there is a winning strategy for Right whether they go first or second

In * = {0|0} there is a winning strategy for whoever goes first

Notation

- $G > 0$ (G is positive) if there is a winning strategy for Left
- $G < 0$ (G is negative) if there is a winning strategy for Right
- $G = 0$ (G is zero) if there is a winning strategy for the second player
- $G * 0$ (G is fuzzy) if there is a winning strategy for the first player
- These can be combined
- $G \geq 0$ means $G > 0$ (a win for Left) or $G = 0$ (a win for second),
  - There is no winning first move for Right
- $G *> 0$ means $G > 0$ (a win for Left) or $G * 0$ (a win for first),
  - There is a winning first move for Left
- $G <* 0$ means $G < 0$ (a win for Right) or $G * 0$ (a win for first)
- $G \leq 0$ means $G < 0$ (a win for Right) or $G = 0$ (a win for second)

Theorem 50

- Each game *G* belongs to one of the outcome classes above
- Proof
- "This is equivalent to the assertion that for each game G, we have either $G \geq 0$ or $G <^* 0$ and either $G \leq 0$ or $G \ {}^*\!> 0$. Suppose this is true for all G^L, G^R. Then if any $G^L \geq 0$ then Left can win by first moving to this G^L then following his strategy for this G^L, Right starting. If not, we have each $G^L <^* 0$ and Right has a winning strategy in G, Left starting. He just sits back and waits for Left to move to some G^L, and then applies his winning strategy (Right starting) in that G^L."

Expanded Proof

- This is equivalent to the assertion that for each game G, we have either $G \geq 0$ or $G <* 0$ and either $G \leq 0$ or $G *> 0$.
- If $G > 0$ (a win for Left) or $G = 0$ (a win for who goes second) then
  - either $G < 0$ or $G = 0$ (a win for Left or Right, whoever goes second:)
  - Or $G * 0$ or $G > 0$ (a win for Left going First or Second)
- Else if $G < 0$ or $G * 0$ (a win for Right or who goes first) then
  - Either $G < 0$ or $G = 0$ (a win for Right going first or second)
  - Or $G * 0$ or $G > 0$ (a win for whoever goes first: Left or Right)

Expanded Proof

- Suppose this is true for all G^L, G^R (all games end so it must eventually be true for some G^L, G^R in every game).
- Then if any $G^L \geq 0$ then Left can win by first moving to this G^L then following his strategy for this G^L, Right starting.
- If G^L then Left moved in G so Right moves First
- If $G^L > 0$ (a win for Left) or $G^L = 0$ (a win for Second) then left will win because they are Left or because they go second.
- Else $G^L < 0$ (a win for right) or $G^L * 0$ (a win for First) then Right will win because they are Right or because they go first.
- If Right moves in G then we are in G^R the same arguments apply by symmetry

The Negative of a Game

- “Since the legal moves of the two players are not necessarily the same, we may obtain a distinct game by reversing the roles of Left and Right throughout *G*. The game so obtained we call the *negative* of G. Inductively, it is the game –G defined by the equation
- $-G = \{-G^R | -G^L\}$
- Obviously negation interchanges the positive and negative games, while the negative of a zero or fuzzy game is another game of the same type.”

Simultaneous Displays

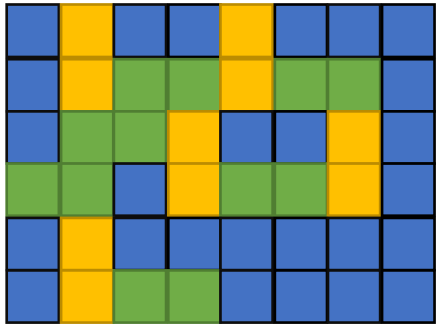
- “Left and Right are given to playing simultaneous displays of games against each other, in the following manner.
- “Each game is placed on a table, and when it is Left’s turn to move he selects one of the component games, and makes any move legal for Left in that game.
- “Then Right selects a component game (possibly the same as that used by Left, possibly not), and makes a move legal for Right in that game.
- “The game continues in this way until some player is unable to move in any of the components, when of course that player loses, according to the normal play convention.”

Sums of Games

- “When games *G* and *H* are played as a simultaneous display in this manner we refer to the compound game as the *disjunctive sum G* + *H* of the two games.” Henceforth referred to as *sums*.
- “In fact it often happens in real-life games that a position breaks up into a disjunctive sum, because it is obvious for some reason that moves made in one part of the position will not affect the other parts. Consider the example of the following game of dominoes suggested by Göran Andersson.

A Game of Dominoes

- “On a rectangular board ruled into squares, the players alternatively place dominoes which cover two adjacent squares, Left being required to place her dominoes vertically, Right horizontally. The dominoes must not overlap, and the last player able to move is the winner.”

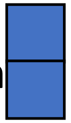
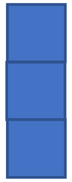
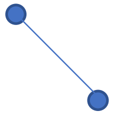


“After a time, the vacant spaces left on the board are usually in several separate regions, and the game becomes a sum of smaller games one for each region.”

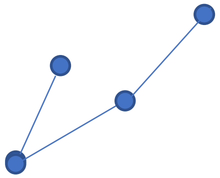
Dominoes: Analysis of Simplest Possibilities

“A region  contains no moves for either player, and so is abstractly the game {|} = 0. 

Such regions can be neglected

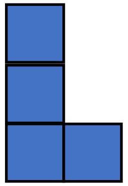
“A region  or  Has just one move for Left (to 0), but none for Right. Its value is therefore {0|} = 1 

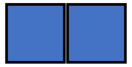
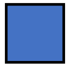
“Similarly the region  Is -2, since it has no move for Left, but moves for Right to 0 and -1, and {|0,-1} = -2

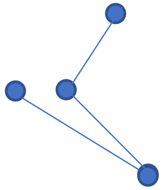


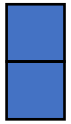
“In General, if a position has no move for Right at any time, and at most *n* successive moves for Left, its value is n, and the value will be –n if we reverse the roles of Left and Right here

Dominoes: Analysis of Simplest Possibilities

“The region  Is more interesting.

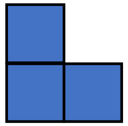
“Left has one (stupid) move to  = -1 and another (more sensible) move to  +  = 0, whereas

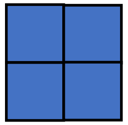


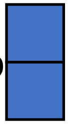
“Right has only one move to  = 1. So the value should be {0, -1|1}, which the diligent reader of the zeroth part of this book* will recognise as ½.”

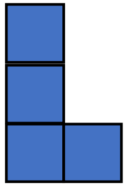
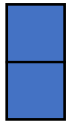
*On Numbers and Games, second edition, John H. Conway

Dominoes: Analysis of Simplest Possibilities

“Values other than numbers can occur in this domino game. The  has value {0|0} = *,
region

“since either player can move to  = 0 (only), while the region  has value {1|-1} since Left

“Moves to  = 1, and Right by symmetry to -1.

“The dominoes position with regions  ,  ,  (only) has the value ½ + 1 – 2 = -½.

“Since this is negative, Right is half-a-move ahead, and can win the game, no matter who starts.

Sums of Simple Games

- “Since it is never legal to move in 0, the game *G* + 0 is essentially the same a *G*, and we write G + 0 = *G*.
- “The game 1 + 1. From the sum 1 + 1, Left can move to 1 + 0 or 0 + 1, both essentially the same as 1. Since Right can never move, we have 1 + 1 = {1, 1|}, and since Left’s two moves are essentially the same, we can simplify this further to 1 + 1 = {1|}. This game we call 2.
- “The game 1 – 1. We write 1 – 1 for the sum 1 + (-1). In this, Left can only move to 0 + -1 = -1 (which is a win for Right), and Right can only move to 1 + 0, a win for left. So neither player will really want to move, and the game is a zero game.
  - In symbols, we have $1 - 1 \equiv \{-1|1\} = 0$
- “The game * + *. In a similar way, $* + * \equiv \{*|*\}$, which, since * is a win for the first player, is a second player win. So we have * + * = 0.

Theorem 51

- *G – G is always a zero game*.
- *Proof*.
- "The moves legal for one player in G become legal for his opponent in – G, and vice versa. So the second player can win $G - G$ by always mimicking her opponent's previous move-if Left moves to G^L in G, Right (as second player) can move to $-G^L$ in $-G$. If she plays in this way, the second player will never be lost of a move in $G - G$.

Theorem 52

- *From $G \geq 0$ and $H \geq 0$, we can deduce $G + H \geq 0$.*
- Proof.
- "The suppositions tell us that if Right starts, Left can win each of G and H. But he can then win $G + H$ by always replying in the component Right moves in, and making the winning reply in this component. In this way, Left cannot be lost for a move in G or H, and so will win the sum.

Theorem 53

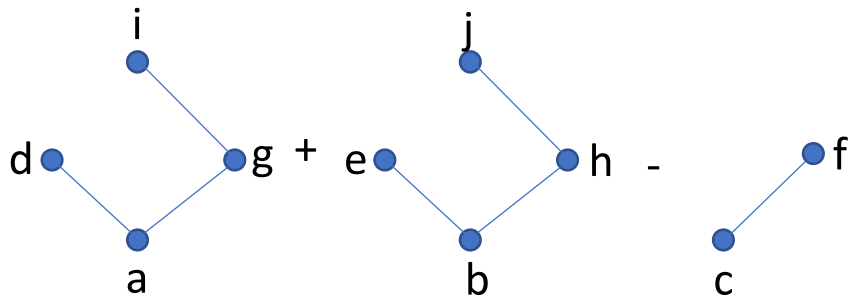
- *If H is a zero game, then $G + H$ has the same outcome as G.*
- *Proof.*
- “Play $G + H$ in exactly the same way as you would in G, never moving in the H component except to reply to an immediately previous move of you component in that game. This rule converts a winning strategy for you in G to one for you in $G + H$, it being understood that the same player starts in both cases.

Theorem 54

- *If H – K is a zero game, then the games G + H and G + K have always the same outcomes*.
- *Proof*.
- “G + H has the same outcome as (G + K) + (H – K) by theorem 53. But this can be written as (G + H) + (K – K) which has the same outcome as G + H as K – K is a zero game.
- “our aim … is to find out who wins sums of various games, so that if *H* – *K* is a zero game, it will not matter if we replace *H* by *K*. So in this case, we shall say that *H* is *equal* to *K*, and write *H* = *K*.
- “We do not distinguish between equal games. By the game 0, we mean also to the games 1 – 1, * + *, and so on.
- “If *G* = *H* then they have the same value (which may not be a number).

The Game ½

- “We define ½ = {0|1}, and verify the equality ½ + ½ = 1.
- Below are the components of the game ½ + ½ - 1 with letter for the names of the various positions.

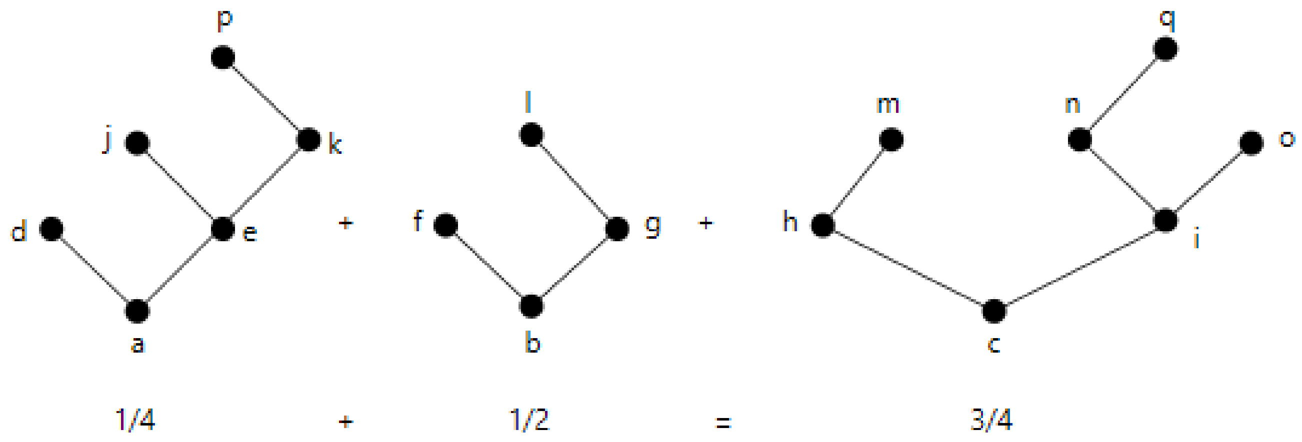


“Initially, we are at position (a,b,c). We consider first what happens if Left starts. He might as we move from a to d, to which Right replies by the move from b to h, then Left can only move from h to j, and Right makes the last move from c to f and wins.

“If Right moves from b to h, Left can reply with a to d, and then wins with h to j as his reply to Rights only move c to f. If in stead Right makes the move c to f, Left can reply a to d, then we have b to h for Right, followed by the winning move h to j. (Note that in all cases we have the same 4 moves a $\rightarrow$ d, b $\rightarrow$ h, h $\rightarrow$ j, c $\rightarrow$ f. This phenomenon often happens.)

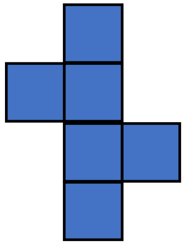
Exercise.

- Taking ¼ as {0|½} and ¾ as {½|1}, give a strategic discussion of the equality ½ + ¼ = ¾.



The game ↑

- The game {0|*} is common enough to deserve a special name, so we call it *up*, and give it a special symbol ↑. Its negative {*|0} – note that * is its own negative, like 0 – is called *down* and given the symbol ↓.
- Since Left wins with the first or second move, ↑ is a positive game.

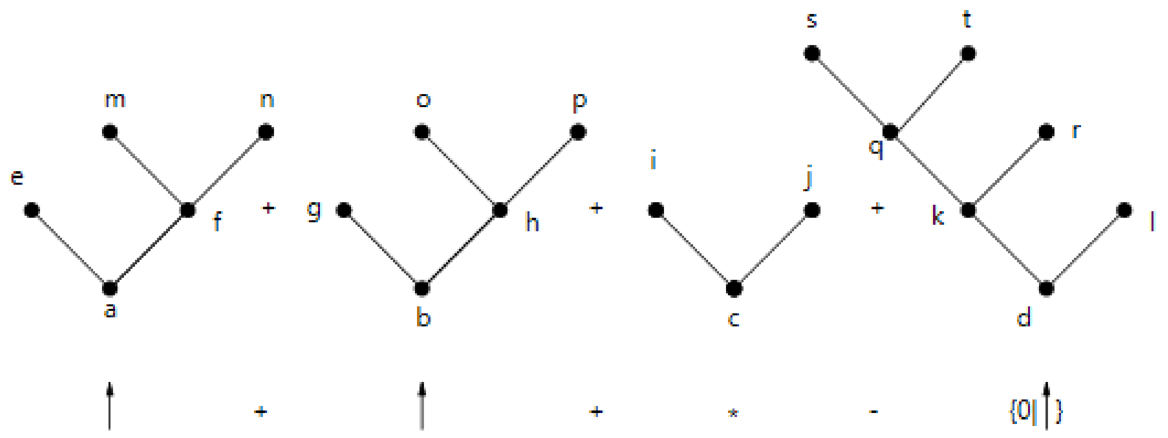
It is the value of the position  in our domino game.

A remarkable Equality

- $\{0|\uparrow\} = \uparrow + \uparrow + *$
- $\{0|\uparrow\} = \{0|*\} + \{0|*\} + \{0|0\}$
- $\{0|*\} + \{0|*\} + \{0|0\} + \{\downarrow|0\} = \{|\}$

The Upstart Equality

The moves a $\to$ f and d $\to$ k lead collectively to the zero position $* + \uparrow + * + \downarrow$, so we can use either as a replay to the other, and then mimic our opponent's moves. So by symmetry we need only consider the moves c $\to$ j, d $\to$ l for Right, and a $\to$ e, c $\to$ I for Left, showing that each has its own counter.



Now the move: c $\to$ j, d $\to$ k lead to position $\uparrow + \uparrow + \downarrow = \uparrow > 0$, and d $\to$ l, c $\to$ I lead to $\uparrow + \uparrow > 0$, so that c $\to$ j and d $\to$ h are bad moves for Right. Similarly, after moves a $\to$ e, b $\to$ h, we have position $* + * + d = d$, and Right wins d, the moves being d $\to$ k, k $\to$ r. In the final case, Right replies to c $\to$ I with a $\to$ f, and then follows on of (f $\to$ m, b $\to$ h), (b $\to$ g, f $\to$ n) and (d $\to$ k, f $\to$ n) and an easy win for Right in each case. So indeed we have $\uparrow + \uparrow + * = \{0|\uparrow\}$

Formally

Construction: if L and R are any two sets of games, there is a game $\{L|R\}$. All games are constructed in this way.

Convention: If $G = \{L|R\}$, we write G^L for the typical element of L, G^R for the typical element of R, and refer to these (respectively) as the Left and Right options of G. Then the *legal moves* in G are for Left, from G to G^L, and for Right to from G to G^R, and we write $G = \{G^L|G^R\}$.

Definition of $G \geq H$, etc.
$G \geq H$ iff (no $G^R \leq H$ and $G \leq$ no H^L). $G \leq H$ iff $H \geq G$. $G * H$ iff neither.
$G *> H$ iff $G \neg\leq H$; $G <* H$ iff $G \neg\geq H$; $G < H$, $G > H$, $G = H$ as usual.

Definition of $G + H$.

$$G + H = \{ G^L + H, G + H^L | G^R + H, G + H^R\}$$

Definition of $-G$.

$$-G = \{-G^R|-G^L\}$$

The Class Pg

“The Class Pg of partisan games form a partially ordered group under addition, with 0 as zero and $-G$ as negative, when considered modulo equality. This group strictly includes the additive group of all numbers. The order-relation is that defined by

$G > H$ iff $G - H$ is won by Left whoever starts
$G < H$ iff $G - H$ is won by Right whoever starts
$G = H$ iff $G - H$ is won by the second player to move, and
$G * H$ iff $G - H$ is won by the first player to move.

“The relation $G * H$ is the relation of incompatibility for this order, meaning that we have no one of $G = H$, $G > H$, $G <$ H. We say then that G and H are confused or that G is *fuzzy* against H.