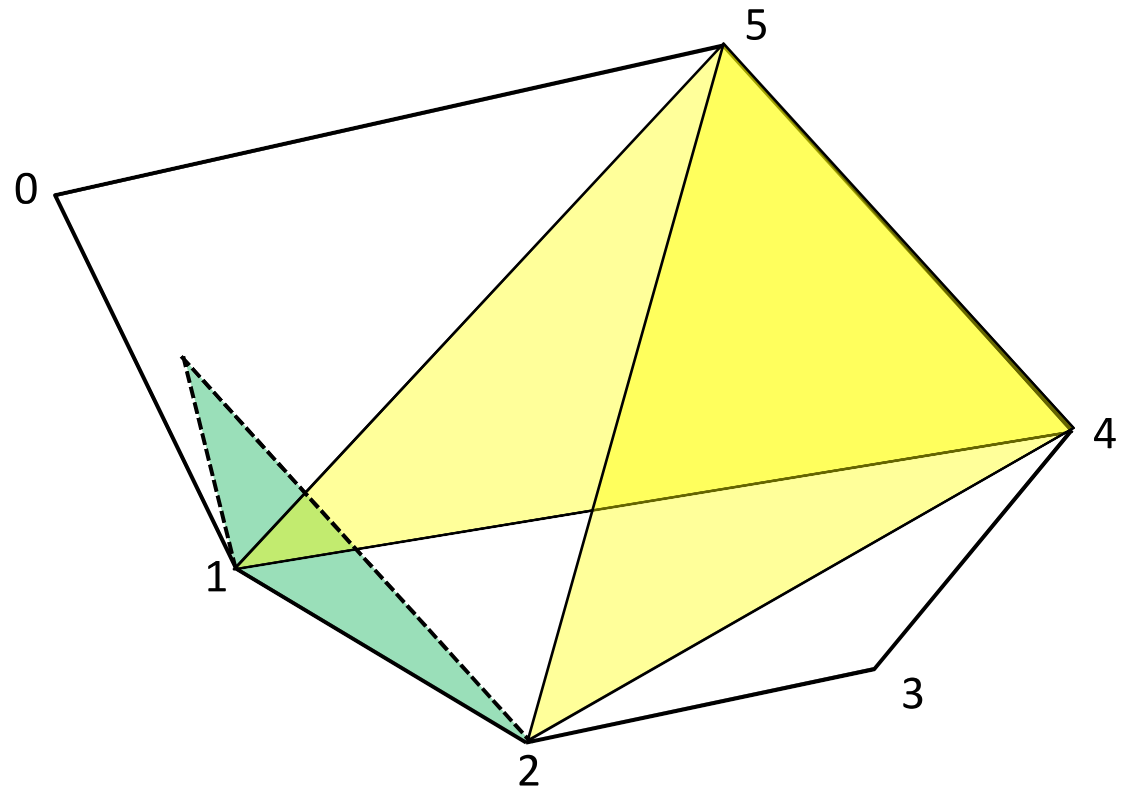
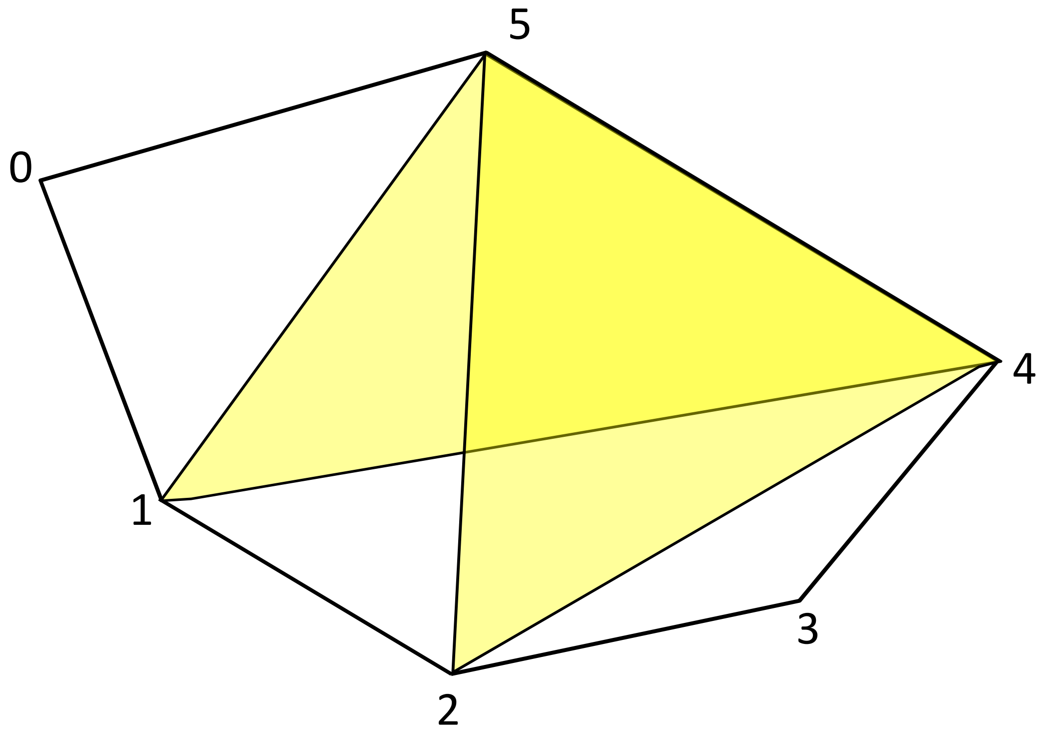
Two matrices of triangle areas

Robin Whitty, Maths Study Group, August 6, 2026



Take two triangles in a polygon



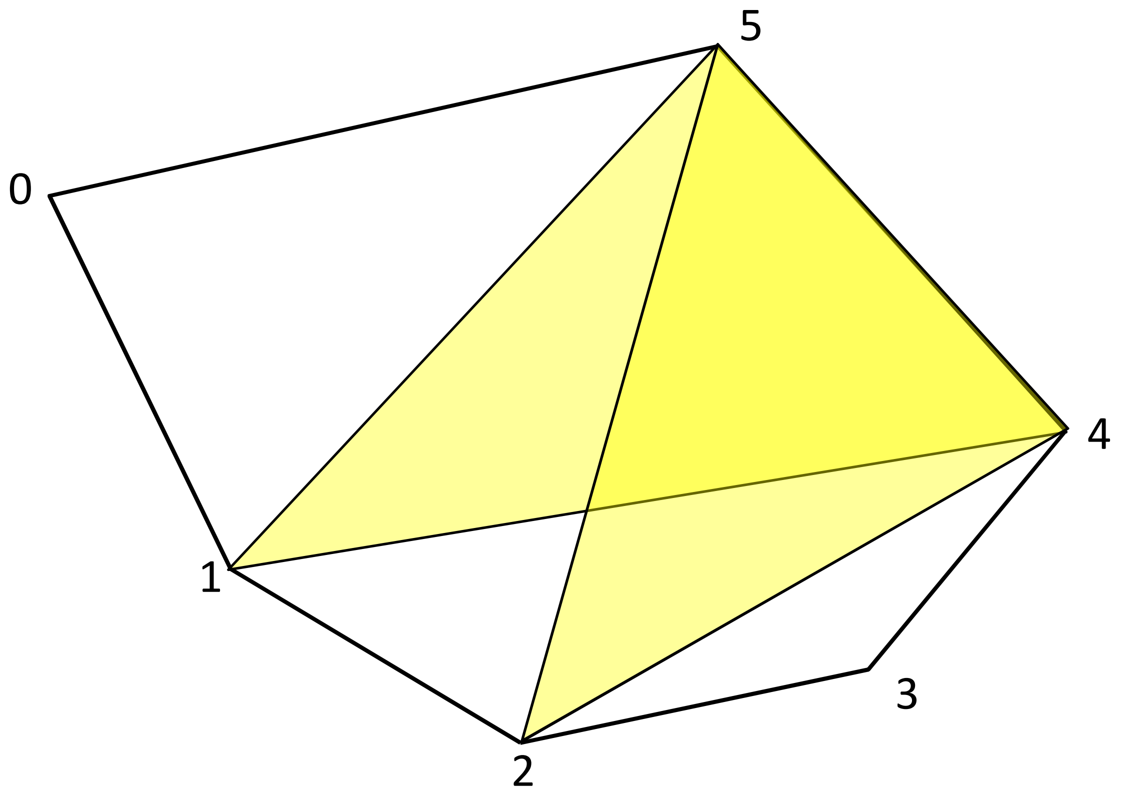
Denote the area of triangle 145 by Δ_{14}

Denote the area of triangle 245 by Δ_{24}

If edges 12 and 45 are parallel then these areas are equal

$$\Delta_{14} = \Delta_{24}$$

In a polygon, take two triangles on *non-parallel edges*



Denote the area of triangle 145 by Δ_{14}

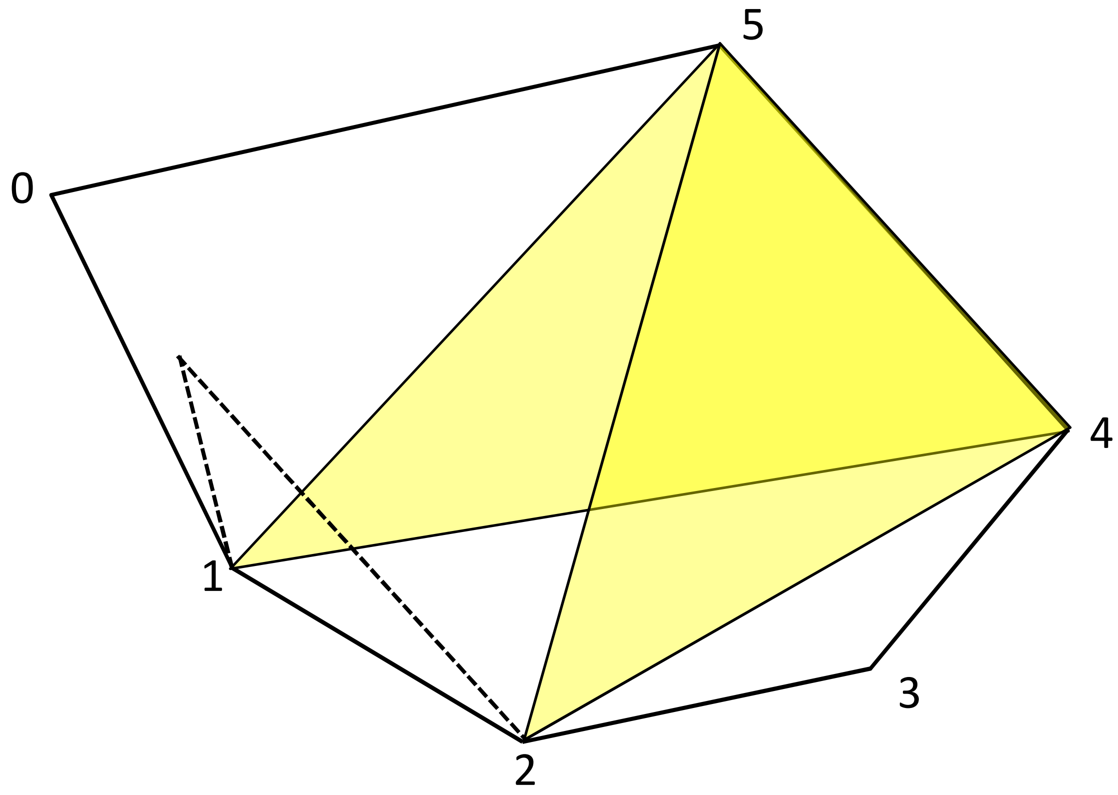
Denote the area of triangle 245 by Δ_{24}

Edges 12 and 45 are not parallel, so

$$\Delta_{14} = \Delta_{24} + \varepsilon$$

What is ε ?

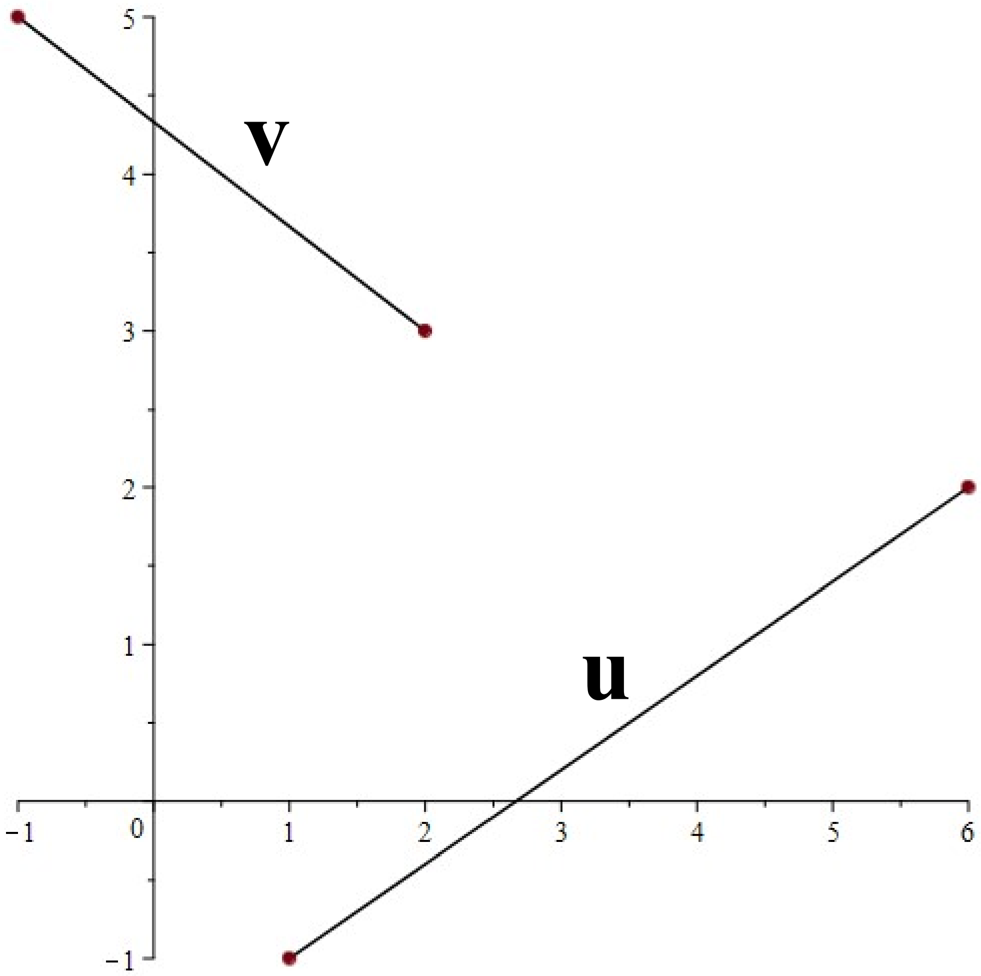
Discrepancy between areas of triangles subtending the same edge from consecutive vertices

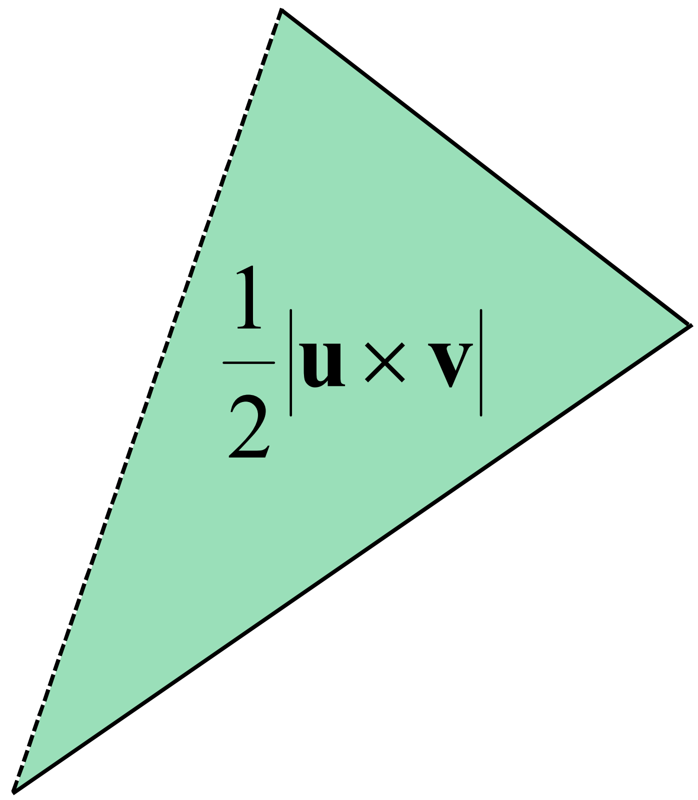


For edges 12 and 45,

$$\Delta_{14} = \Delta_{24} + \text{area of triangle formed from edges 12 and 45}$$

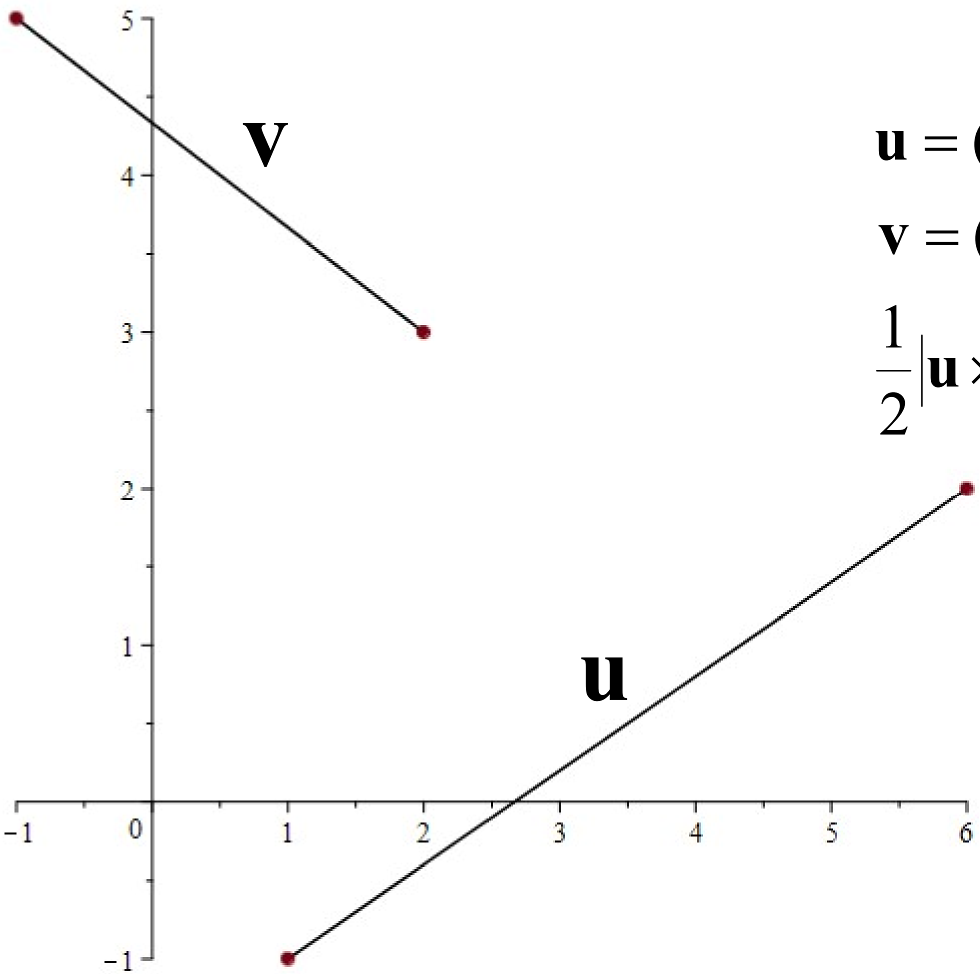
Triangle areas are (magnitudes of) cross products





$$\left|\mathbf{u}\times\mathbf{v}\right| = u_x v_y - u_y v_x$$

Cross products from coordinates

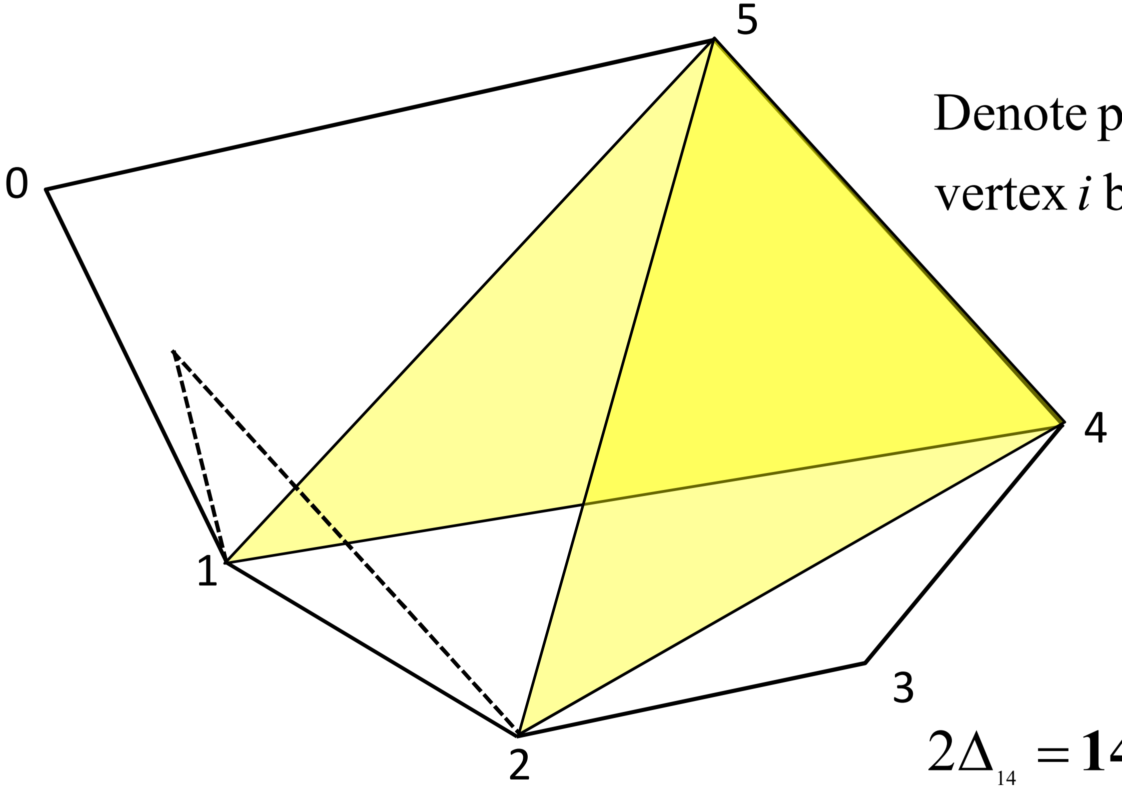


$$\mathbf{u} = (6,2) - (1,-1) = (5,3)$$

$$\mathbf{v} = (-1,5) - (2,3) = (-3,2)$$

$$\frac{1}{2}\left|\mathbf{u}\times\mathbf{v}\right| = \frac{1}{2}(5\times 2 - 3\times -3) = \frac{19}{2}$$

Confirming our area discrepancy (algebraically)



Denote position vector (coordinates) of vertex i by v_i

$$\mathbf{14} = v_4 - v_1$$

$$\mathbf{15} = v_5 - v_1$$

$$\mathbf{24} = v_4 - v_2$$

$$\mathbf{25} = v_5 - v_2$$

$$\mathbf{45} = v_5 - v_4$$

$$2\Delta_{14} = \mathbf{14} \times \mathbf{45} = (v_4 - v_1) \times (v_5 - v_4)$$

$$2\Delta_{24} = \mathbf{24} \times \mathbf{45} = (v_4 - v_2) \times (v_5 - v_4)$$

$$\begin{aligned} 2(\Delta_{14} - \Delta_{24}) &= ((v_4 - v_1) - (v_4 - v_2)) \times (v_5 - v_4) \\ &= (v_2 - v_1) - (v_5 - v_4) \\ &= \mathbf{12} \times \mathbf{45} \end{aligned}$$

A matrix of triangle areas

Why do I care about discrepancies between areas of triangles in a polygon?

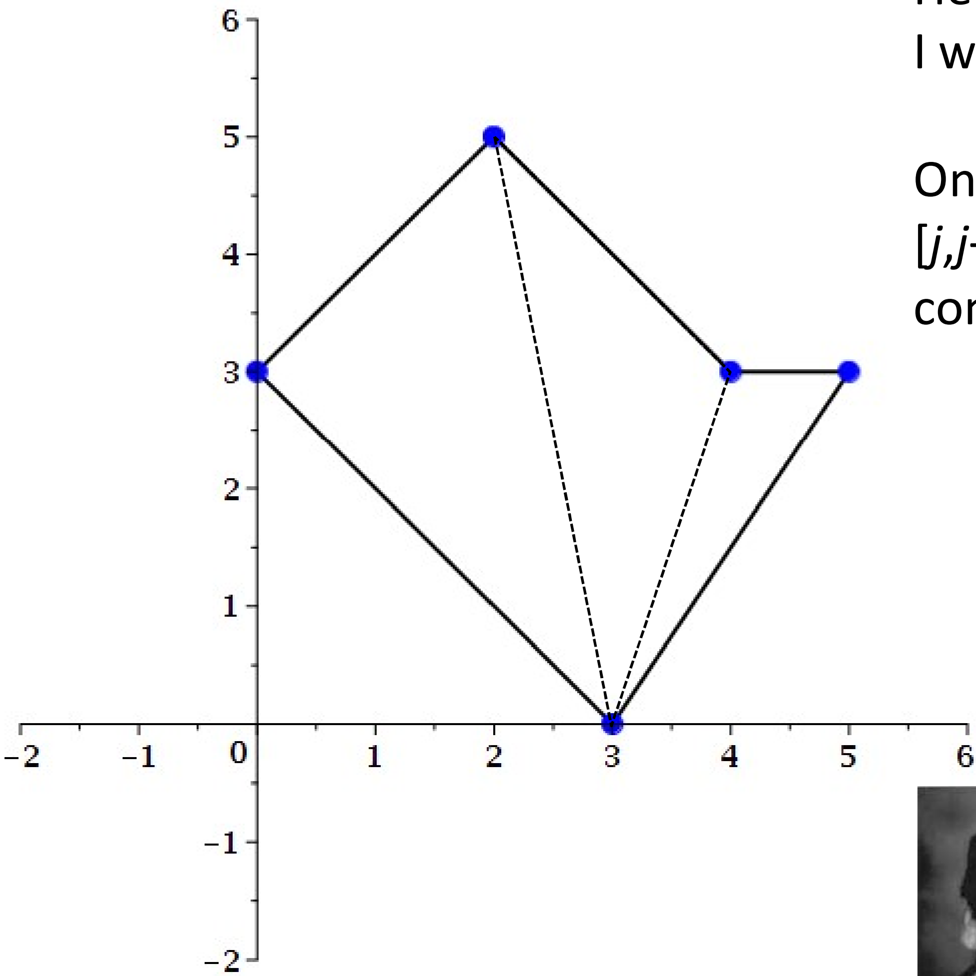
I don't, directly.

But all the areas subtending all the edges in a polygon form a matrix, call it Δ. The linear algebra of Δ is useful (to me).

All the discrepancies between all the areas also forms a matrix, call it Γ. It has a linear algebra which is a bit 'cleaner' than for Δ.

But the two matrices are directly connected (as we have seen). So I hope to learn about Δ (e.g. its eigenvectors) by looking at Γ.

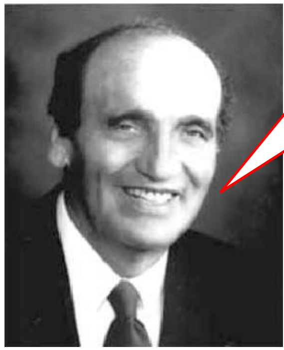
Vertex-edge subtending triangles

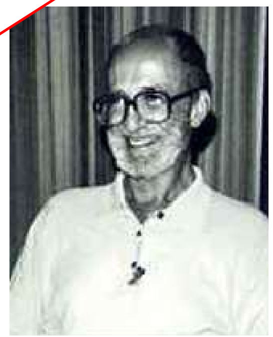


Here is a polygon on 5 vertices which I will call 0, …, 4.

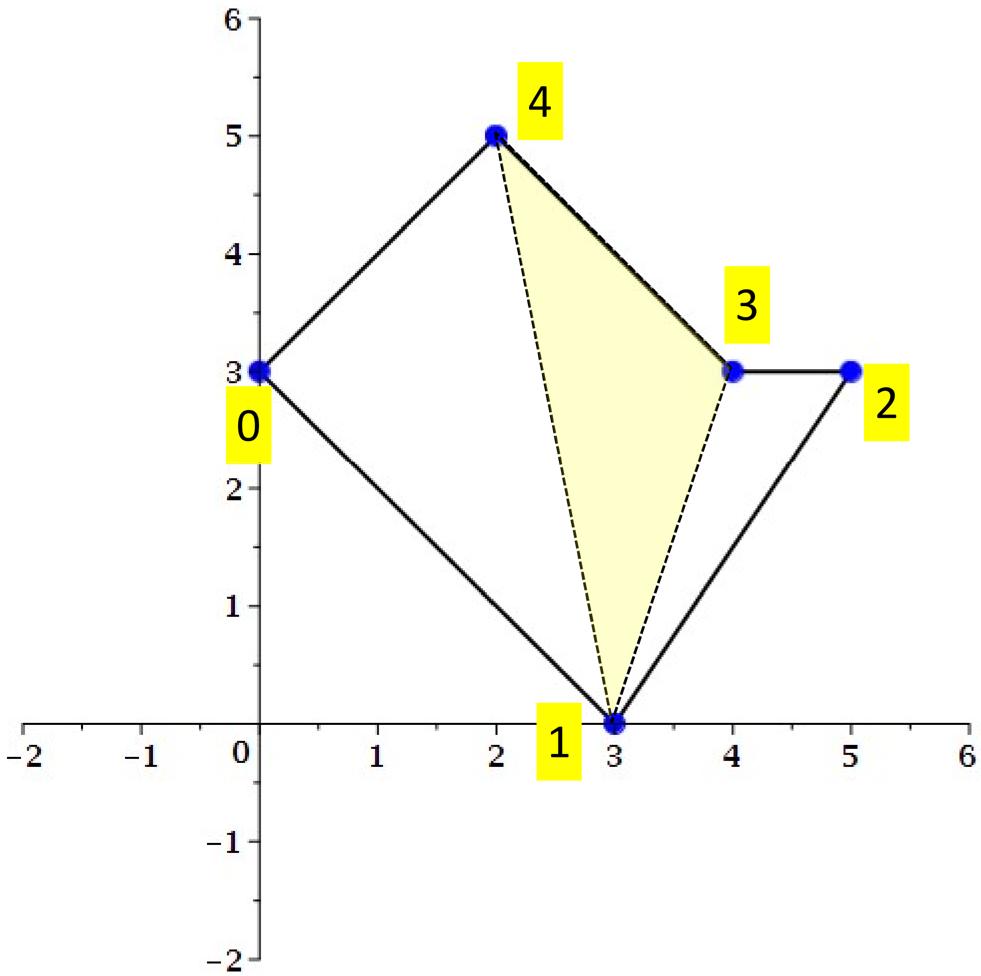
On vertex *i*, form a triangle to edge [*j*,*j*+1] (square brackets to avoid confusion with coordinate pairs)

We share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.





Properties of subtended triangle areas



The area of the triangle subtending edge [*j*,*j*+1] from vertex *i* will be denoted by $\Delta_{i,j}$.

The triangle shown here has area $\Delta_{1,3}$.

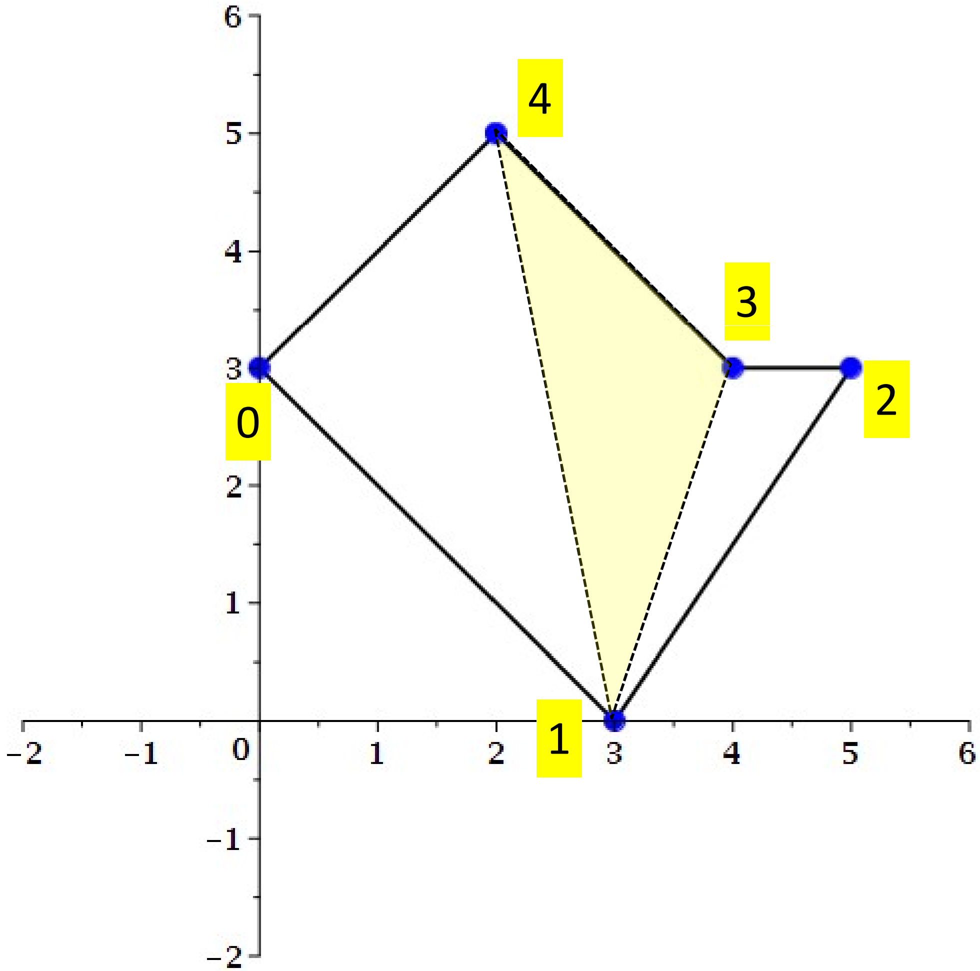
Some or all of the area counted by $\Delta_{i,j}$ may lie outside of the polygon. E.g. $\Delta_{2,3}$ is all area outside the polygon.

Clockwise triangles have negative area. E.g. $\Delta_{2,3} = -1$.

3 collinear vertices give zero area. E.g. $\Delta_{0,2} = 0$.

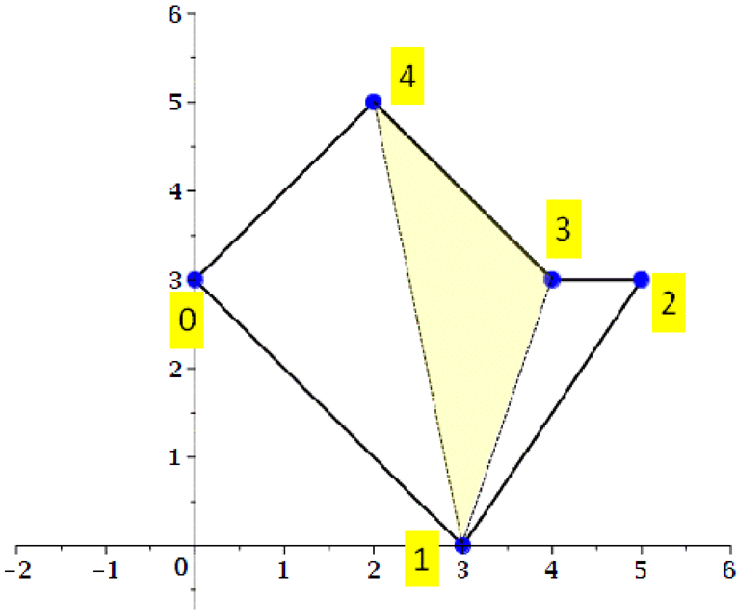
The triangles areas matrix Δ

Taking $\Delta_{i,i} = 0$ and $\Delta_{i-1,i} = 0$ the totality of all $\Delta_{i,j}$ for an n vertex polygon form an $n \times n$ matrix, which we call the *triangle areas matrix*, denoted by Δ.



$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of the triangle areas matrix



$$\begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 24 & 6 & \frac{45}{4} & 30 & 61 \\ \frac{285}{4} & 45 & -6 & -\frac{3}{2} & \frac{47}{2} \\ 24 & \frac{349}{4} & -5 & 30 & -4 \\ 24 & 71 & -\frac{7}{4} & 30 & 9 \\ -\frac{15}{2} & 45 & \frac{39}{4} & 51 & 34 \end{bmatrix}$$

$$\Delta \qquad\qquad \Delta^2$$

The rows of Δ sum to the area of the polygon (equivalent to the Shoelace formula).

Δ has rank 3 (Forbes, A.D., Whitty, R.W. and Moorhouse, T., “Solution 324.2 – Triangles”, *M500*, No. 327, 2026, pp. 1 – 8).

With *A* the area of the polygon and Tr being the trace function (sum of diagonal elements), the eigenvalues of Δ are:

$$\overbrace{0, \ldots, 0}^{n-3}, A, -\frac{A}{2} \pm \frac{1}{2}\sqrt{2\mathrm{Tr}(\Delta^2) - 3A^2}.$$

It’s the last two eigenvalues which are mysterious to me.

Eigenvalues and eigenvectors

A complex number λ is an eigenvalue of square matrix X if, for some complex row vector v,

$$\begin{aligned} vX &= \lambda v && v \text{ is a left eigenvector} \\ Xv^T &= \lambda v^T && v^T \text{ is a right eigenvector} \end{aligned}$$

E.g. $X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -2 & -2 & -2 \end{pmatrix}$

Eigenvalues are 0 and $1 \pm 2i$.

Right eigenvectors are

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1+i)/2 \\ (-2+i)/2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1-i)/2 \\ (-2-i)/2 \\ 1 \end{pmatrix}.$$

Δ's all-1s eigenvector

If all rows of a square matrix X have the same sum s then s is an eigenvalue of X and the all-ones column vector is a right eigenvector.

$$X\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}=\begin{pmatrix}s\\ \vdots\\ s\end{pmatrix}=s\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

$$\begin{bmatrix}0 & \frac{15}{2} & 0 & 4 & 0\\ 0 & 0 & \frac{3}{2} & 4 & 6\\ \frac{15}{2} & 0 & 0 & -1 & 5\\ 6 & \frac{3}{2} & 0 & 0 & 4\\ 6 & \frac{13}{2} & -1 & 0 & 0\end{bmatrix} \&* \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{23}{2}\\ \frac{23}{2}\\ \frac{23}{2}\\ \frac{23}{2}\\ \frac{23}{2}\end{bmatrix}$$

Δ's two complex eigenvectors

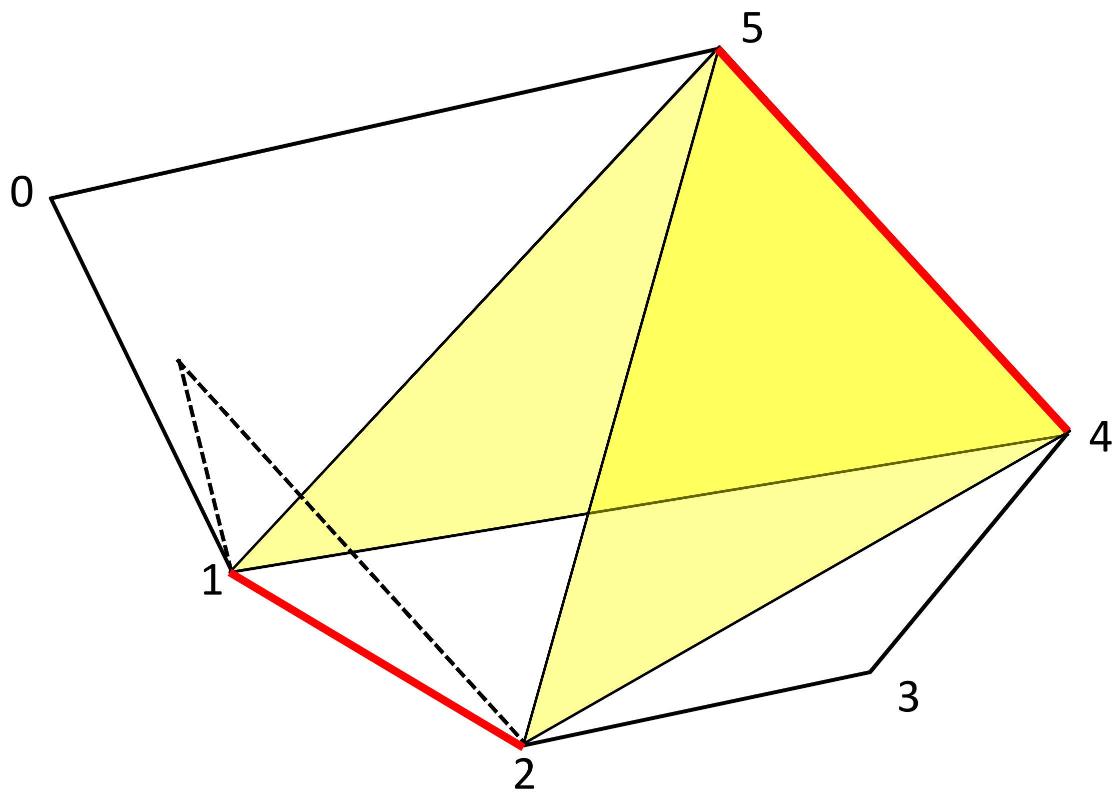
How to write down (in closed form) a vector satisfying this equation…

$$\begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = -\frac{23}{4} + \frac{1}{2}\sqrt{2\times 128 - 3\times\left(\frac{23}{2}\right)^2} \times \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

$$-\frac{23}{4} + \frac{i}{4}\sqrt{563}$$

… ideally reflecting the geometry of the polygon!

Introducing: the matrix Γ



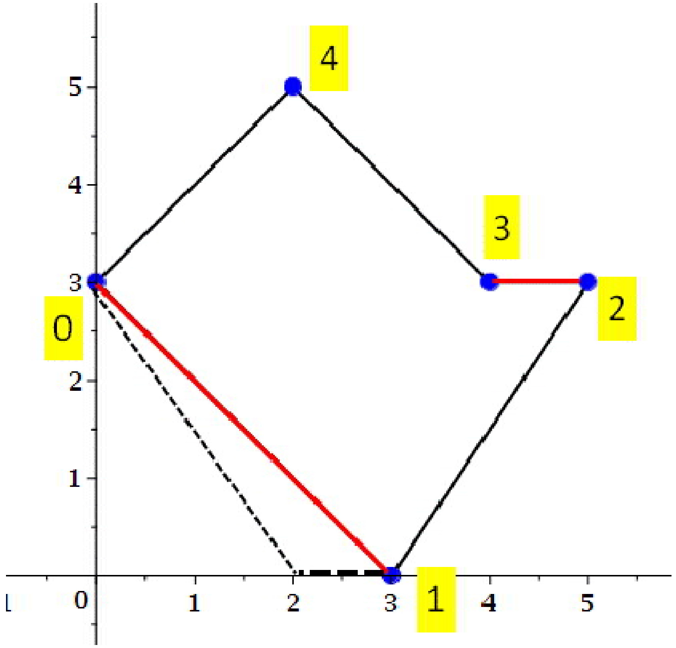
For each edge $[i,i+1]$ of an n-vertex polygon, take the cross product with each edge $[j,j+1]$, in turn. Record half its magnitude as element i,j in an $n \times n$ matrix called Γ:

$$\Gamma_{i,j} = \frac{1}{2}\left|(v_{i+1} - v_i) \times (v_{j+1} - v_j)\right|,$$

the vertices being taken modulo n.

I shall refer to Γ as the discrepancy matrix of the polygon although it is really just another matrix of triangle areas.

Properties of the discrepancies matrix Γ



$$
\Gamma \quad
\begin{array}{c|c}
 & \begin{array}{ccccc} [0,1] & [1,2] & [2,3] & [3,4] & [4,0] \end{array} \\
\begin{array}{c} [0,1] \\ [1,2] \\ [2,3] \\ [3,4] \\ [4,0] \end{array} &
\begin{bmatrix}
0 & \frac{15}{2} & -\frac{3}{2} & 0 & -6 \\
-\frac{15}{2} & 0 & \frac{3}{2} & 5 & 1 \\
\frac{3}{2} & -\frac{3}{2} & 0 & -1 & 1 \\
0 & -5 & 1 & 0 & 4 \\
6 & -1 & -1 & -4 & 0
\end{bmatrix}
\end{array}
$$

Γ is skew-symmetric.

The rows of Γ sum to zero.

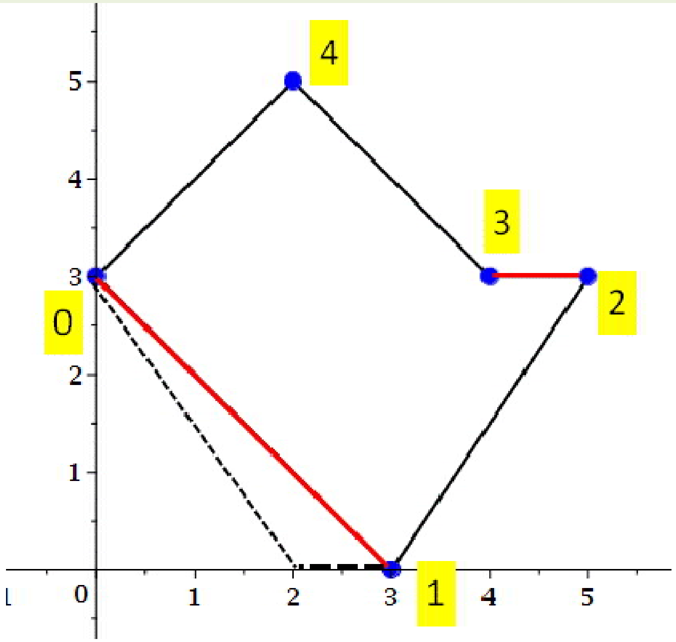
Γ has rank 2 (by Cramer's rule).

With Tr being the trace function, the eigenvalues of Γ are:

$$\overbrace{0,\ldots,0}^{n-2},\pm\sqrt{\frac{1}{2}\mathrm{Tr}\left(\Gamma^2\right)}$$

The last two eigenvalues are still mysterious to me. But they are a lot 'cleaner' (the polygon area is disappeared from the picture).

Going between Γ and Δ



$$\Gamma_{i,j} = \text{discrepancy between areas}$$

$$\Delta_{i,j} \text{ and } \Delta_{i+1,j}$$

$$\text{I.e. } \Gamma_{i,j} = \Delta_{i,j} - \Delta_{i+1,j}$$

$$\text{E.g., } \Delta_{0,2} = 0, \Delta_{1,2} = \frac{3}{2} \text{ and so } \Gamma_{0,2} = -\frac{3}{2}$$

$$\Gamma \quad \begin{array}{cc} & \begin{array}{ccccc} [0,1] & [1,2] & [2,3] & [3,4] & [4,0] \end{array} \\ \begin{array}{c} [0,1] \\ [1,2] \\ [2,3] \\ [3,4] \\ [4,0] \end{array} & \begin{bmatrix} 0 & \frac{15}{2} & -\frac{3}{2} & 0 & -6 \\ -\frac{15}{2} & 0 & \frac{3}{2} & 5 & 1 \\ \frac{3}{2} & -\frac{3}{2} & 0 & -1 & 1 \\ 0 & -5 & 1 & 0 & 4 \\ 6 & -1 & -1 & -4 & 0 \end{bmatrix} \end{array}$$

$$\Delta \quad \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix} \end{array}$$

Going from Δ to Γ algebraically

For all $i, j, \Gamma_{i,j} = \Delta_{i,j} - \Delta_{i+1,j}$

So we can write

$$\Gamma = (I - C_n)\Delta,$$

where C_n is the matrix of the cycle on n elements

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{I - C_5} \&* \underbrace{\begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix}}_{\Delta} = \underbrace{\begin{bmatrix} 0 & \frac{15}{2} & -\frac{3}{2} & 0 & -6 \\ -\frac{15}{2} & 0 & \frac{3}{2} & 5 & 1 \\ \frac{3}{2} & -\frac{3}{2} & 0 & -1 & 1 \\ 0 & -5 & 1 & 0 & 4 \\ 6 & -1 & -1 & -4 & 0 \end{bmatrix}}_{\Gamma}$$

Going from Γ to Δ

We would like to go the other way: start with the much simpler Γ and derive Δ. Unfortunately we can't use matrix multiplication to do this because matrix multiplication preserves the lowest rank of its operands and Γ has lower rank than Δ.

For all $i, j, \Gamma_{i,j} = \Delta_{i,j} - \Delta_{i+1,j}$

So $\Delta_{i,j} = \Gamma_{i,j} + \Delta_{i+1,j}$

Together with the fact that $\Delta_{i,i+1} = \Gamma_{i,i+1}$ this

gives us a recursive construction of Δ from Γ

$$\begin{bmatrix} 0 & \frac{15}{2} & -\frac{3}{2} & 0 & -6 \\ -\frac{15}{2} & 0 & \frac{3}{2} & 5 & 1 \\ \frac{3}{2} & -\frac{3}{2} & 0 & -1 & 1 \\ 0 & -5 & 1 & 0 & 4 \\ 6 & -1 & -1 & -4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix}$$

Going from Γ to Δ recursively

$$\begin{bmatrix} 0 & G_{1,2} & G_{1,3}+G_{2,3} & G_{1,4}+G_{2,4}+G_{3,4} & 0 \\ 0 & 0 & G_{2,3} & G_{2,4}+G_{3,4} & G_{1,2}+G_{1,3}+G_{1,4} \\ G_{1,2} & 0 & 0 & G_{3,4} & G_{1,3}+G_{2,3}+G_{1,4}+G_{2,4} \\ G_{1,2}+G_{1,3} & G_{2,3} & 0 & 0 & G_{1,4}+G_{2,4}+G_{3,4} \\ G_{1,2}+G_{1,3}+G_{1,4} & G_{2,3}+G_{2,4} & G_{3,4} & 0 & 0 \end{bmatrix}$$

5×5 matrix Δ derived from Γ

$$\begin{bmatrix} 0 & G_{1,2} & G_{1,3}+G_{2,3} & G_{1,4}+G_{2,4}+G_{3,4} & G_{1,5}+G_{2,5}+G_{3,5}+G_{4,5} & 0 \\ 0 & 0 & G_{2,3} & G_{2,4}+G_{3,4} & G_{2,5}+G_{3,5}+G_{4,5} & G_{1,2}+G_{1,3}+G_{1,4}+G_{1,5} \\ G_{1,2} & 0 & 0 & G_{3,4} & G_{3,5}+G_{4,5} & G_{1,3}+G_{2,3}+G_{1,4}+G_{2,4}+G_{1,5}+G_{2,5} \\ G_{1,2}+G_{1,3} & G_{2,3} & 0 & 0 & G_{4,5} & G_{1,4}+G_{2,4}+G_{3,4}+G_{1,5}+G_{2,5}+G_{3,5} \\ G_{1,2}+G_{1,3}+G_{1,4} & G_{2,3}+G_{2,4} & G_{3,4} & 0 & 0 & G_{1,5}+G_{2,5}+G_{3,5}+G_{4,5} \\ G_{1,2}+G_{1,3}+G_{1,4}+G_{1,5} & G_{2,3}+G_{2,4}+G_{2,5} & G_{3,4}+G_{3,5} & G_{4,5} & 0 & 0 \end{bmatrix}$$

6×6 matrix Δ derived from Γ

These are not geometrical objects! They use only the Γ→ Δ recurrence rule and the properties of Γ (zero row sums, skew-symmetric). So they don't quite transfer eigenvalue properties between Γ and Δ as they stand.