A Very Basic Introduction to Hopf Algebras

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1 Introduction

Heinz Hopf, one of the pioneers of Algebraic topology, first introduced these algebras in connection with the homology of Lie groups in 1939. Later, in the 1960s Milnor introduced the Steenrod algebra, the algebra of cohomology operations, which was another example of a Hopf algebra.

More recently the study of these algbras has gained pace because of their applications in Physics as quantum groups, renormalisation and non-comutative geometry.

In the late 1970s Rota introduced Hopf algebras into combinatorics and there is now a well established research field know as combinatorial Hopf algebras. There are many examples of familiar combinatorial objects which can be given the structure of a Hopf algebra, we look at a few simple examples at the end of this work.

To date these ideas do not seem to have penetrated computer science to any great extent.

The first part of this work is heavilly based on seminar notes by Rob Ray of North Carolina State University.

2 Vector spaces and Tensor Products

Vectors can be added and multiplied by scalars. Examples: geometric vectors, matrices, polynomials. Scalars are members of the ground field $\mathbb{R}$ or $\mathbb{C}$, could also be $\mathbb{Z}_2$ or other finite field; for generality the ground field will be refered to as $\mathbb{F}$ here.

Let V and W be vector spaces with typical elements $v_i \in V$ and $w_j \in W$ then the tensor product of V and W is a vector space $V \otimes W$ with typical element $v_i \otimes w_j$. The tensor product also has the following linearity properties,

$$(a_i v_i) \otimes w_k + (a_j v_j) \otimes w_k = (a_i v_i + a_j w_j) \otimes w_k$$

and

$$v_i \otimes (b_k w_k) + v_i \otimes (b_l w_l) = v_i \otimes (b_k w_k + b_l w_l)$$

for any vectors $v_i, v_j \in V$ and $w_k, w_l \in W$ and any scalars $a_i, a_j, b_k b_l \in \mathbb{F}$.

This means that if $\{v_1, v_2, \dots v_m\}$ and $\{w_1, w_2, \dots w_n\}$ are bases for V and W respectively then $\{v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, \dots v_m \otimes w_n\}$ is a basis for $V \otimes W$.

As an example, suppose $V = \text{Span}(v_1, v_2)$ then $V \otimes V \text{Span}(v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2)$. Another way of looking at this is to think of v_1 and v_2 as column vectors,

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

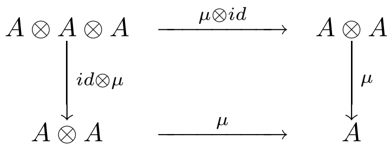
then the tensor product $V \otimes V$ has the basis,

$$v_1 \otimes v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; v_1 \otimes v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; v_2 \otimes v_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } v_2 \otimes v_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

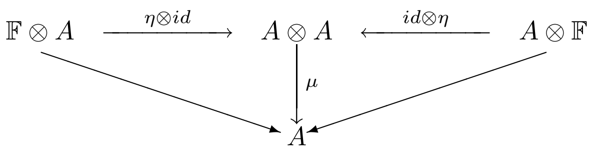
Finally here, we have the relation $\mathbb{F} \otimes V = V$ for any vector space V (over $\mathbb{F}$). The idea here is that any field $\mathbb{F}$ can be thought of as a vector space over itself with single base element 1. The isomorphism is given by identifying the basis elements v_i of V with the elements $1 \otimes v_i$ in $\mathbb{F} \otimes V$. Notice that in particular we have that $\mathbb{F} \otimes \mathbb{F} = \mathbb{F}$.

3 Algebras and Coalgebras

In this context an algebra over $\mathbb{F}$ is defined as a vector space A (over $\mathbb{F}$) with a linear multiplication map $\mu : A \otimes A \longrightarrow A$, and a unit, given by a linear unit map $\eta : \mathbb{F} \longrightarrow A$, such that the following diagrams commute:



and



The first of these diagrams ensures that the algebra is associative. The second diagram expresses the properties of the unit. Let $1_A = \eta(1)$, then the second diargam says $\mu(1_A \otimes x) = \mu(x \otimes 1_A) = x$ for all $x \in A$.

Simple example, polynomials in one variable $\mathbb{C}[x]$, this is not a finite vector space, the basis 'vectors' are $1, x, x^2, x^3, \ldots$. Multiplication is as you would expect, for example:

$$\mu(1 \otimes x) =, \; \mu(x \otimes x) = x^2, \; \mu(x \otimes x^2) = x^3, \; \mu(x^2 \otimes x^3) = x^5, \ldots$$

Since multiplication is a linear map we also have, $\mu(3x \otimes 4x^2) = 12x^3$ for example. Also polynomial multiplication is associative. Finally, the unit map is simply $\eta(1) = 1$ and the required relations clearly hold.

The advantage of defining the concept of an algebra using diagrams, in the style of Category theory, is that it is simply to 'dualise' our definitions. To do this we simply reverse all the arrows. So we get the following definition of a coalgebra.

A coalgebra is a vector space C together with two linear maps, comultiplication, $\Delta : C \longrightarrow C \otimes C$ and counit (sometimes called augmentation) $\epsilon : C \longrightarrow \mathbb{F}$ such that the following diagrams commute.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Big\downarrow \Delta & & \Big\downarrow id\otimes\Delta \\ C \otimes C & \xrightarrow{\Delta\otimes id} & C \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccccc} & \swarrow & C & \searrow & \\ & & \Big\downarrow \Delta & & \\ \mathbb{F} \otimes C & \xleftarrow{\epsilon\otimes id} & C \otimes C & \xrightarrow{id\otimes\epsilon} & C \otimes \mathbb{F} \end{array}$$

The first of these diagrams expresses the so called coassociative property of the comultiplication.

We can use $\mathbb{C}[x]$ as an example again. The comultiplication on basis elements is given by,

$$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i}$$

where $x^0 = 1$. So for example,

$$\Delta(x^2) = 1 \otimes x^2 + x \otimes x + x^2 \otimes 1$$

It it not too difficult to see that this is coassociative, consider a small example,

$$\begin{aligned} (id \otimes \Delta)(\Delta(x^2)) &= 1 \otimes (1 \otimes x^2 + x \otimes x + x^2 \otimes 1) \\ &\quad + x \otimes (1 \otimes x + x \otimes 1) + x^2 \otimes (1 \otimes 1) \\ &= 1 \otimes 1 \otimes x^2 + 1 \otimes x \otimes x + 1 \otimes x^2 \otimes 1 + x \otimes 1 \otimes x \\ &\quad + x \otimes x \otimes 1 + x^2 \otimes 1 \otimes 1 \end{aligned}$$

and the other way around gives

$$\begin{aligned} (\Delta \otimes id)\Delta(x^2) &= (1 \otimes 1) \otimes x^2 + (1 \otimes x + x \otimes 1) \otimes x \\ &\quad + (1 \otimes x^2 + x \otimes x + x^2 \otimes 1) \otimes 1 \\ &= 1 \otimes 1 \otimes x^2 + 1 \otimes x \otimes x + x \otimes 1 \otimes x + 1 \otimes x^2 \otimes 1 \\ &\quad + x \otimes x \otimes 1 + x^2 \otimes 1 \otimes 1 \end{aligned}$$

These are the same except for some reordering of terms.

The counit map is given by,

$$\epsilon(x^n) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

As another example consider $\mathbb{C}G$, the group algebra of a group G. The basis vectors here are the group elements $g \in G$. A typical element of $\mathbb{C}G$ has the form,

$$z_1 g_1 + z_2 g_2 + \cdots$$

where the $z_i \in \mathbb{C}$ are complex scalars and $g_i \in G$. This space is also both an algebra and a coalgebra. Multiplication on basis elements is simply the group product, $\mu(g_i \otimes g_j) = g_i g_j$ with the identity element as unit; $\eta(1) = e$. These maps can then be extended to the whole vector space by invoking linearity.

The coalgebra structure is given by the comultiplication $\Delta(g) = g \otimes g$ and counit map $\epsilon(g) = 1$ for all $g \in G$ and then extened by linearity again. Notice that this coalgebra is cocommutative, that is the following diagram commutes,

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \tau} \\ \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \end{array}$$

The map τ here swaps the factors in $C \otimes C$, that is $\tau(a \otimes b) = b \otimes a$. Notice that this diagram is the dual of the diagram one would draw to express the fact that a multiplication μ was commutative.

In any coalgebra an element g which satisfies $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ is called group-like.

Note, in computer science coalgebras are defined slightly differently, the linear structures are not used and the maps are just defined on sets.

3.1 Morphisms and Bialgebras

A bialgebra is a vector space that is both an algebra and a coalgebra. Further, the algebra and co algebra structures must be compatible. The compatibility is ensured by requiring either of the following equivalent conditions.

1. Δ and ϵ must be algebra morphisms.

2. μ and η must be coalgebra morphisms.

The equivalence of these conditions becomes clear if we express them as commutative diagrams. In general a linear map f between two algebras A and B is a morphism if the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ \Big\downarrow{\scriptstyle \mu_A} & & \Big\downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow{f} & B \end{array}$$

Now the comultiplication is a linear map between A and $A \otimes A$, so the condition that Δ is an algerbra morphism becomes the condition that the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\Delta \otimes \Delta} & A \otimes A \otimes A \otimes A \\ \downarrow \mu & & \downarrow \mu_{A \otimes A} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array}$$

This is the same diagram that one would get by demanding that μ is a coalgebra morphism. However, we must be a little careful here, the map $\mu_{A \otimes A}$ involves the twist map τ,

$$\mu_{A \otimes A} = (\mu \otimes \mu) \circ (id \otimes \tau \otimes id)$$

This is most easily seen by writting the multiplication μ as $\cdot$ using infix notation, so the compatibility condition becomes,

$$\Delta(a_1 \cdot a_2) = \Delta(a_1) \cdot \Delta(a_2)$$

for all elements a_1, a_2 in the bialgebra. This is still not absolutely clear, consider the group algebra example $\mathbb{C}G$, from above, we have

$$\Delta(g_1 g_2) = g_1 g_2 \otimes g_1 g_2$$

from the definition of the coproduct and also

$$\Delta(g_1)\Delta(g_2) = (g_1 \otimes g_1)(g_2 \otimes g_2) = g_1 g_2 \otimes g_1 g_2$$

as required.

The vector space of polynomials $\mathbb{C}[x]$ as defined above, is not a bialgebra because the multiplication and comultiplications given above are not compatible. However, if we modify the definition of the coproduct we can make it into a bialgebra.

4 Hopf Algebra - the Antipode

Finally we are (almost) in a position to define a Hopf algebra. A Hopf algebra is a bialgebra with an antipode.

To explain what an antipode is we need to look at linear maps from a coalgebra C say to an algebra A. Given two such maps $f, h : C \longrightarrow A$ we can get a new map from C to A as follows,

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes h} A \otimes A \xrightarrow{\mu} A$$

The linear map obtained in this way is called the convolution of f and h, written $f \star h$, In terms of composition of maps we can write,

$$f \star h = \mu \circ (f \otimes h) \circ \Delta$$

Now an antipode S is an endomorphism of a bialgebra H, that is a linear map from H to itself, which satisfies,

$$id_H \star S = S \star id_H = \eta \circ \epsilon$$

That is, an antipode is the convolution inverse to the identity. If it exists it is necesarilly unique.

On group-like elements the antipode gives the inverse,

$$\begin{aligned}(id_H \star S)(g) = \mu \circ (id_H \otimes S) \circ \Delta(g) &= \eta \circ \epsilon(g) \\ \mu \circ (id_H \otimes S)(g \otimes g) &= \eta(1) \\ \mu(g \otimes S(g)) &= 1_H\end{aligned}$$

Hence, we can conclude that a group algebra $\mathbb{C}G$ is a Hopf algebra with antipode $S(g) = g^{-1}$.

5 Examples

5.1 Free Associative Algebras

Let $A = \{s_1, s_2, \ldots, s_k\}$ be a finite set of letters—an alphabet. Now denote the set of all possible words by $A^* = \{1, s_1, s_1s_1, \ldots, s_1s_2, \ldots\}$. Finally consider the vector space $\mathbb{F}A^*$ whose basis elements are all the elements of A^*, this is the free associative (non-commutative) algebra on A. Multiplcation of basis elements is simply concatenation and the unit elements 1 is the empty word.

Comultiplication can be defined as,

$$\Delta(s_n) = \sum_{i=0}^{n} s_i \otimes s_{n-i}$$

For example,

$$\Delta(s_1) = 1 \otimes s_1 + s_1 \otimes 1, \qquad \Delta(s_2) = 1 \otimes s_2 + s_1 \otimes s_1 + s_2 \otimes 1, \ldots$$

This would seem to require a total order for the elements of A.

Notice that we have only given the comultiplications for elements of A, for other basis elements we use the morphism formula so that the comultiplication is automatically compatible with multiplication. For example,

$$\begin{aligned}\Delta(s_1s_1) &= \Delta(s_1)\Delta(s_1) \\ &= (1 \otimes s_1 + s_1 \otimes 1)(1 \otimes s_1 + s_1 \otimes 1) \\ &= (1 \otimes s_1s_1) + 2(s_1 \otimes s_1) + (s_1s_1 \otimes 1)\end{aligned}$$

In fact it is straightforward to show that,

$$\Delta(s_1^n) = \sum_{i=0}^{n} \binom{n}{i} s_1^i \otimes s_1^{n-i}$$

and of course this is what we should have used for the comultiplication in the polynomial algebra $\mathbb{C}[x]$, to get a bialgebra. Notice that if A has only one element then the free associative algebra on A and the polynomial algebra in one variable are the same.

The counit must satisfythe relations,

$$\begin{aligned} 1 \otimes w &= (\epsilon \otimes id)\Delta(w) \\ w \otimes 1 &= (id \otimes \epsilon)\Delta(w) \end{aligned}$$

for all w in the algebra. These are easily satisfied by,

$$\epsilon(w) = \begin{cases} 1, & \text{if } w = 1 \\ 0, & \text{if } w \neq 1 \end{cases}$$

which is an algebra morphism.

To find the antipode we use the following lemma.

In a graded bialgebra an antipode can be defined recursively, so any graded bialgebra is necessarilly a Hopf algebra.

A graded bialgebra is one that can be split up into pieces each with a different grade or degree. Formally, a bialgebra B is graded if $B = \bigoplus_{n\geq 0} B_n$ with $B_0 = \mathbb{F}$ and $\mu(B_i, B_j) \subseteq B_{i+j}$ and $\Delta(B_n) \subseteq \bigoplus_{i+j=n} B_i \otimes B_j$.

Here B_i is a homogeneous piece of B containing all the elements of degree i. Multiplying homogeneous elements must produce a homogeneous element whose degree is the sum of the degrees of the factors. The coproduct must split a homogeneous element into homogeneous factors whose degrees add to the degree of the original element. Clearly the free associative algebra is graded in this sense, with the degree given by the length of the word.

Now the recursive definition of the antipode is given by, $S(1) = 1$, then for any $x \in B_n, n \geq 1$,

$$S(x) = -\sum_{i=1}^{m} S(y_i) \cdot z_i$$

where

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^{m} y_i \otimes z_i$$

In a free associative algebra this works very nicely, for example,

$$\Delta(s_1) = s_1 \otimes 1 + 1 \otimes s_1$$

so that,

$$S(s_1) = -S(1)s_1 = -s_1$$

For the second letter we get,

$$\Delta(s_2) = s_2 \otimes 1 + s_1 \otimes s_1 + 1 \otimes s_2$$

so that,

$$S(s_2) = -S(s_1)s_1 - S(1)s_2 = s_1 s_1 - s_2$$

For $s_1 s_1$ we get,

$$\Delta(s_1 s_1) = s_1 s_1 \otimes 1 + 2 s_1 \otimes s_1 + 1 \otimes s_1 s_1$$

so that,

$$S(s_1 s_1) = -2S(s_1)s_1 - S(1)s_1 s_1 = s_1 s_1$$

and so on.

To see how this works first look at the counit on a graded bialgebra, assume that for $x \in B_n$, $n \geq 1$,

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^{m} y_i \otimes z_i$$

as above. Then, from the relation satisfied by the counit we have,

$$(id \otimes \epsilon)\Delta(x) = x \otimes \epsilon(1) + \sum_{i=1}^{m} y_i \otimes \epsilon(z_i) = x \otimes 1$$

From this we can conclude that in a graded bialgebra $\epsilon(1) = 1$ and $\epsilon(z) = 0$ for all $z \in B_i$ with $i \geq 1$. Now we can substitute our formula for $\Delta(x)$ into the relation for the antipode,

$$\begin{aligned}
\mu \circ (S \otimes id) \circ \Delta(x) &= \eta \circ \epsilon(x) \\
\mu \circ (S \otimes id)\big(x \otimes 1 + \sum_{i=1}^{m} y_i \otimes z_i\big) &= 0 \\
\mu\big(S(x) \otimes 1 + \sum_{i=1}^{m} S(y_i) \otimes z_i\big) &= 0 \\
S(x) + \sum_{i=1}^{m} S(y_i) z_i &= 0 \\
S(x) &= -\sum_{i=1}^{m} S(y_i) z_i
\end{aligned}$$

Notice that this Hopf algebra is cocommutative.

There is another Hopf algebra defined on this space. If take the product as the shuffle product of words and the coproduct that splits words into all possible pairs of prefixes and suffixes then we get a commutative Hopf algebra. I think these two Hopf algebras must be dual to each other.

5.2 Posets

The following is taken from a paper by Ehrenborg, that Robin gave me. The partially ordered sets here all have a minimal element 0 and a maximal element 1 and have a finite number of elements. Consider the vector space $\mathcal{I}$ whose basis elements are isomorphism classes of these posets. The multiplication in this algebra is given by the

Cartesian product of posets. If P and Q are two posets then their product is $P \times Q$. The order relation in this new poset is given by,

$$(x, y) \leqslant (z, w) \Leftrightarrow x \leqslant z \text{ and } y \leqslant w$$

where $x, z \in P$ and $y, w \in Q$. The unit is the poset with just one element. The algebra is graded by the height of the posets, that is the length of the longest chain in the poset; from 0 to 1.

The comultiplication is given by the following,

$$\Delta(P) = \sum_{x \in P} [0, x] \otimes [x, 1]$$

where $[x, y]$ is the interval $[x, y] = \{z \in P : x \leqslant z \leqslant y\}$. Notice that $[x, y]$ is a poset with minimal element x and maximal element y.

The antipode is closely related to the Möbius function μ of the poset. Let $\phi : \mathfrak{I} \longrightarrow \mathbb{F}$ be a linear functions whose value on the basis elements is 1; $\phi(P) = 1$. Notice that this function just counts the number of basis elements in a general element. Now the recurrence relation gives,

$$S(P) = -\sum_{x \in P \setminus \{1\}} S([0, x]) \times [x, 1]$$

Applying the map ϕ to this gives the recurrence relation for the Möbius function, so that,

$$\mu(P) = \phi(S(P))$$

5.3 Rooted Trees

There are several ways to make the space of rooted trees into a Hopf algebra, we briefly look at the one described by Grossman and Larson. Consider the vector space whose basis consists of finite rooted trees.

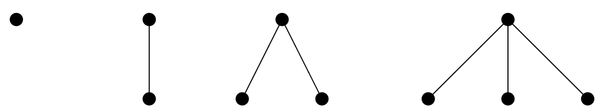


Figure 1: Some basis elements.

The number of edges in the tree gives a grading on the space with the tree consisting of a single node as the only element with grade 0. This is the unit element in the algebra.

To define the coproduct we first look at the following operations. Suppose t is a rooted tree then let $B_-(t)$ denote the set of trees obtained from t by deleting the root node and its edges. Next we introduce the $B_+(\{t_1, t_2, \ldots, t_n\})$ this operation takes a set of rooted trees $\{t_1, t_2, \ldots, t_n\}$ and forms a new rooted tree by adding a new root

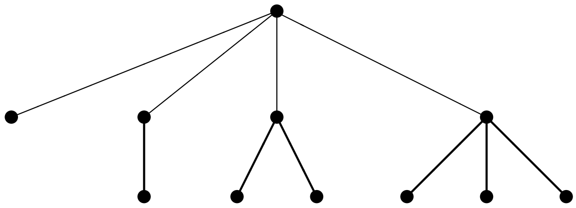


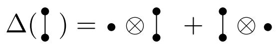
Figure 2: For B_- remove the root and thin lines, for B_+ put them back!

node and joining it to the roots of the trees $t_1, t_2, \ldots, t_n$. Clearly these two operations are mutually inverse.

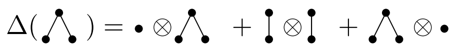
Now the coproduct can be defined as,

$$\Delta(t) = \sum_{X \subseteq B_-(t)} B_+(X) \otimes B_+(\overline{X})$$

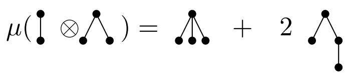
here X and $\overline{X}$ partition $B_-(t)$, that is $X \cap \overline{X} = \text{Ø}$ and $X \cap \overline{X} = B_-(t)$. For example,



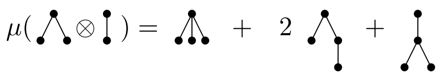
and also



To compute the Grossman-Larson product $\mu(t_1 \otimes t_2)$ we takes every tree in $B_-(t_1)$ and attach it to a node in t_2, then sum over all possibilities. Again this is best illustrated by an example,



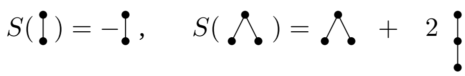
Notice that this product is non-commutative, for example, To prove that these definitions form a Hopf algebra is non-trivial.



Since the algebra is graded we can find the antipode recursively. As usual we have that the antipode of the unit is just the unit, $S(\bullet) = \bullet$. The recurrence relation for trees with 1 or more edges is,

$$S(t) = -\sum_{X \subset B_-(t)} \mu\Big(S\big(B_+(X)\big) \otimes B_+(\overline{X})\Big)$$

Notice that the sum now ranges over proper subset of $B_-(t)$ here. So for example we have,



and so forth. Notice that for trees whose roots have a single child, that is $t_p = B_+(t)$ for some tree t we have that, $S(t_p) = -t_p$.

5.4 Primitive Elements

A primitive element in a Hopf algebra is one that satisfies the relation,

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

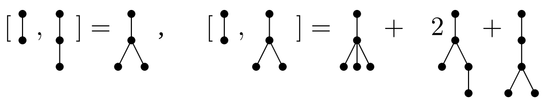
The set of all such elements in a Hopf algebra form a Lie algebra with commutator as Lie bracket. That is, $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$. It will be easier to write the product μ using infix notation . here, $[x, y] = x.y - y.x$. Now commutators in associative algebras automatically satisfy the Jacobi identity, so all we have to check here is that the set of primitive elements is closed under the bracket operation. So assume that x and y are two primitive elements,

$$\begin{aligned}
\Delta([x, y]) &= \Delta(x.y - y.x) \\
&= \Delta(x).\Delta(y) - \Delta(y).\Delta(x) \\
&= (x \otimes 1 + 1 \otimes x).(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y).(x \otimes 1 + 1 \otimes x) \\
&= x.y \otimes 1 + y \otimes x + x \otimes y + 1 \otimes x.y \\
&\qquad -y.x \otimes 1 - x \otimes y - y \otimes x - 1 \otimes y.x \\
&= (x.y - y.x) \otimes 1 + 1 \otimes (x.y - y.x)
\end{aligned}$$

In the Grossman-Larson algebra of rooted trees the primitive elements are trees of the form $t_p = B_+(t)$, that is trees whose root has a single child. Then we get,

$$\Delta(t_p) = B_+(t) \otimes B_+(\text{Ø}) + B_+(\text{Ø}) \otimes B_+(t) = t_p \otimes \bullet + \bullet \otimes t_p$$

So for example we have,



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Rob Ray's seminar notes can be found at,

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and,

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or via his home-page at, http://www4.ncsu.edu/ rcray/

The material on free associative algebras comes from,

> M.D. Crossley, The Steenrod algebra and other copolynomial Hopf Algebras *Bull. lond. math. soc.* **32**:609–614, 2000. Available online at http://www-maths.swan.ac.uk/staff/mdc/publns/copolyst.pdf (last visit 17/1/05)

The paper on posets mentioned is,

> R. Ehrenborg, On posets and Hopf algebras *Avances in Mathematics* **119**:1–25, 1996 Available online at, http://www.ms.uky.edu/ jrge/Papers/Hopf.pdf (last visit 17/1/05)

This also contains the stuff on gradings.

There is lots of stuff on rooted trees, for example,

> H. Berland and B. Owren, Algebraic structures on ordered rooted trees and their significance to Lie group integrators. Preprint 3/2003, Norwegian University of Science and Technology Tronnddheim.
> Available online at, http://www.math.ntnu.no/preprint/numerics/2003/N3-2003.ps (last visit 17/1/05)

and

> M. E. Hoffman, Combinatorics of Rooted Trees and Hopf Algebras *arXiv:math.CO/0201253*, 2003 Available via http://arxiv.org/abs/math.CO/0201253 (last visit 17/1/05)

and many others.

The classic text on the subject is,

> M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

But I haven't managed to get hold of this.

Also of interest: The following site is for a meeting on Combinatorial Hopf algebras held in Sept. 2004.

http://www.pims.math.ca/birs/workshops/2004/04w5011/index.html