THREE PRIMES

The Hardy–Littlewood circle method is used to prove Vinogradov’s theorem: every sufficiently large odd integer is the sum of three primes

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Background

We shall closely follow Modern Prime Number Theory by T. Estermann, CUP, 1961.

We adopt the convention that the variable $p$ with or without a subscript always ranges over the primes. Let us also get some function definitions out of the way:

$$e(x) = e^{2\pi ix},$$

$$\mu(v) = \begin{cases} \prod_{p|v} (-1) & \text{if } v \text{ is square-free,} \\ 0 & \text{otherwise} \end{cases},$$

the Möbius function;

$$\phi(v) = \sum_{0<h\leq v, \gcd(h,v)=1} 1,$$

Euler’s phi function;

$$c_q(v) = \sum_{0<h\leq q, \gcd(h,q)=1} e\left(\frac{hv}{q}\right),$$

Ramanujan’s sum.

Observe that $c_q(v)$ is just the sum of the $v$-th powers of the primitive $q$-th roots of 1.

**Lemma 1**  
(i) For any $X$ and any integer $v$, $\int_{X}^{X+1} e(vx)dx = 0$ if $v \neq 0$ and 1 if $v = 0$.

(ii) The Möbius function and Euler’s phi function are multiplicative. Ramanujan’s sum is multiplicative over $q$.

(iii) If $\gcd(v,q) = 1$, then $c_q(v) = c_q(1)$.

(iv) If $\gcd(v,q) = 1$, then $c_q(v) = \mu(q)$.

**Proof**  Property (i) is fundamental to a lot of what follows. Its proof is straightforward. If $\gcd(q_1, q_2) = 1$, then

$$c_{q_1}(v)c_{q_2}(v) = \sum_{h_1, h_2} e\left(v\left(\frac{h_1}{q_1} + \frac{h_2}{q_2}\right)\right) = \sum_{h_1, h_2} e\left(\frac{v(h_1q_2 + h_2q_1)}{q_1q_2}\right) = c_{q_1q_2}(v)$$

since $e\left(\frac{h_1q_2 + h_2q_1}{q_1q_2}\right)$ runs over the primitive $(q_1q_2)$-th roots of 1. That takes care of Ramanujan’s sum. The other two are elementary number theory. This proves (ii).

Part (iii) is obvious (I think). For (iv), we can assume $v = 1$ by (iii). If $k \geq 1$, $c_{p^k}(1)$ is the sum of the $(p^k)$-th roots of 1 minus the sum of the $(p^{k-1})$-th roots of 1; that is, $-1$ when $k = 1$ and 0 otherwise. Part (iv) follows by multiplicativity. $\square$

In its simplest form the prime number theorem states that $\pi(x) \sim x/(\log x)$. For a more accurate version we define the **logarithmic sum**,

$$\text{ls}(x) = \sum_{2 \leq m \leq x} \frac{1}{\log m}.$$  

This is like the logarithmic integral except that it is a sum, and it is easily seen that the difference between the two is bounded: $\text{ls}(x) - \text{li}(x) = O(1)$. 

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Theorem 1 (T19 : Theorem 19 in Estermann’s book)

\[ \pi(x) = \text{ls}(x) + O \left( x \exp \left( -\frac{\sqrt{\log x}}{200} \right) \right). \]

This (with li(x) instead of ls(x)) was proved by de la Vallée Poussin in 1898ish. Like the li(x) form it is a very good approximation. That complicated error term is ultimately superior to \( x/(\log x)^m \) for any fixed positive \( m \) but it is worse than \( x^{1-\delta} \) for any fixed \( \delta > 0 \) however small. The best error term for the prime number theorem is still the 45-year-old result of H.-E. Richert (1967). There exists a positive constant \( C_1 \) such that

\[ \pi(x) = \text{li}(x) + O \left( x \exp \left( -\frac{C_1 (\log x)^{3/5}}{\log \log x} \right) \right), \]

proved by establishing that the Riemann zeta function \( \zeta(\sigma + it) \) has no zeros with \( \sigma \geq 1 - C_2 (\log t)^{-2/3} (\log \log t)^{-1/3} \) for some positive constant \( C_2 \). The Holy Grail of the subject is of course to extend the zero-free region westwards all the way up to the line \( \sigma = 1/2 \), with the consequent improvement in the error term of Theorem 1 to \( O(\sqrt{x \log x}) \). This is the Riemann Hypothesis. In the other direction Littlewood showed that

\[ \pi(x) - \text{li}(x) = \Omega_{\pm} \left( \frac{x^{1/2} \log \log x}{\log x} \right). \]

We will also need the prime number theorem for arithmetic progressions.

Theorem 2 (T55) Let \( u > 0 \). Let \( q \leq (\log x)^u \) and \( \gcd(h, q) = 1 \). Then the number of primes \( p \leq x, p \equiv h \pmod{q} \) is given by

\[ \pi(x; q, h) = \frac{\text{ls}(x)}{\phi(q)} + O \left( x \exp \left( -\frac{\sqrt{\log x}}{200} \right) \right), \]

where the constant implied by the \( O \) notation is independent of \( q \) and \( h \).

The proof requires the severe restriction on \( q \) to guarantee uniformity with respect to \( q \), which is vital for our application. Uniformity with respect to \( h \) is ‘trivial’. A long-standing problem in this area is to extend the range of \( q \). Write

\[ E(x, q) = \max_{\gcd(h, q) = 1} \left| \frac{\pi(x; q, h) - \pi(x)}{\phi(q)} \right|. \]

Then Theorems 1 and 2 state that for \( q \leq (\log x)^{15} \), say,

\[ E(x, q) = O \left( x \exp \left( -\frac{\sqrt{\log x}}{200} \right) \right). \]

The Elliott–Halberstam conjecture is that on average one can relax the condition on \( q \): for every \( \theta < 1 \) and \( A > 0 \) there exists a constant \( C_3 > 0 \) such that

\[ \sum_{1 \leq q \leq x^\theta} E(x, q) \leq \frac{C_3 x}{(\log x)^A}. \]

Bombieri and Vinogradov showed that the Elliott–Halberstam conjecture holds for \( \theta < 1/2 \).
Three primes

Vinogradov proved in 1937 that every sufficiently large odd integer can be represented in the form \( p_1 + p_2 + p_3 \). Previously, in 1923, Hardy and Littlewood had shown that this is true if there exists a number \( \delta < 3/4 \) such that none of Dirichlet’s \( L \)-functions has zeros in the half-plane \( \Re z > \delta \). More recently, in 1989 Chen & Wang showed that the three primes representation holds (unconditionally) for odd \( n > 10^{43000} \).

Let \( r(n) \) denote the number of solutions of \( n = p_1 + p_2 + p_3 \); that is,

\[
  r(n) = \sum_{\substack{n = p_1 + p_2 + p_3 \text{ prime} \ p_1, p_2, p_3 \geq 2}} 1.
\]

Repetitions are allowed and order is relevant; so \( r(11) = 6 \) because \( 11 = 2 + 2 + 7 = 2 + 7 + 2 = 3 + 3 + 5 = 3 + 5 + 3 = 5 + 3 + 3 = 7 + 2 + 2 \). Let

\[
  \rho(n) = \sum_{\substack{n = m_1 + m_2 + m_3 \text{ } m_1, m_2, m_3 \geq 2}} \frac{1}{\log m_1} \frac{1}{\log m_2} \frac{1}{\log m_3}.
\]

Ultimately we want the following.

**Theorem 3** Let

\[
  S(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi^3(q)} c_q(n).
\]

Then

\[
  r(n) = S(n) \rho(n) + O \left( \frac{n^2}{(\log n)^4} \right).
\]

The proof will occupy the next three sections of these Notes.

For now we observe that the thing being summed in Theorem 3, \( \mu(q)c_q(n)/\varphi^3(q) \), is multiplicative as a function of \( q \) (see Lemma 1). Moreover, \( \mu(1) = 1, \mu(p) = -1 \) and \( \mu(p^k) = 0 \) for \( k \geq 2 \). So \( S(n) \) has the simple product form:

\[
  S(n) = \prod_p \left( 1 - \frac{c_p(n)}{(p-1)^3} \right).
\]

If \( n \) is even, then \( c_2(n) = e(n/2) = 1, S(n) = 0 \), and Theorem 3 doesn’t say anything interesting. On the other hand, when \( n \) is odd we have \( c_2(n) = -1, |c_p(n)| \leq p - 1 \) for odd \( p \) and hence

\[
  S(n) = 2 \prod_{p > 2} \left( 1 - \frac{c_p(n)}{(p-1)^3} \right) \geq 2 \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \geq 2 \prod_{m=2}^{\infty} \left( 1 - \frac{1}{m^2} \right) = 1.
\]

Furthermore we can estimate \( \rho(n) \). For \( n \geq 6 \), the number of terms in the sum for \( \rho(n) \) is

\[
  \sum_{m_1=2}^{n-4} (n - m_1 - 3) = \frac{1}{2} (n - 4)(n - 5)
\]

and each term is at least \( 1/(\log n)^3 \). Therefore

\[
  \rho(n) > \frac{1}{(\log n)^3} \sum_{m_1=2}^{n-4} (n - m_1 - 3) > \frac{n^2}{3(\log n)^3}.
\]
for sufficiently large \( n \). Thus we have our desired result:

\[
r(n) > \frac{n^2}{3(\log n)^3} + O \left( \frac{n^2}{(\log n)^4} \right).
\]

**The computation of \( r(n) \)**

For \( v \geq 0 \), let

\[
f(x, v) = \sum_{p \leq v} e(px).
\]

Then

\[
f(x, v)^3 = \sum_{p_1 \leq v} \sum_{p_2 \leq v} \sum_{p_3 \leq v} e((p_1 + p_2 + p_3)x)
\]

and

\[
r(n) = \int_{x_0}^{x_0 + 1} f(x, n)^3 e(-nx)dx.
\]

Curiously, this formula actually works, at least for small \( n \). Putting it into Mathematica gives this table.

| \( n \) | 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 |
| \( r(n) \) | 1 3 3 4 6 6 9 6 6 10 9 12 12 12 12 19 12 21 15 21 18 30 15 30 12 |

But our main task is to find a non-trivial general lower bound for \( r(n) \) by proving Theorem 3. We shall estimate the integral in (1). Henceforth we will assume tacitly that \( n \) is sufficiently large. For convenience we fix

\[
x_0 = \frac{(\log n)^{15}}{n},
\]

which we want to be small—and it will be provided \( n \) is sufficiently large.

It happens that \( f(x, n)^3 \) is small unless \( x \) is near a rational number with a small square-free denominator. Even for the tiny value \( n = 601 \) we can clearly see the spikes at 0, 1/2, 1/3, 2/3, 1/6 and 5/6 as well as lesser peaks at \( j/10 \) for \( j = 1, \ldots, 9, j \neq 5 \); but there are none at 1/4, 3/4, 1/8, 3/8, 5/8 and 7/8. It turns out that the half-width of the six most prominent spikes is about 0.001, and if we integrate \( f(x, 601)^3 e(-nx) \) over just the intervals \( a \pm 0.001, a = 0, 1/2, 1/3, 2/3, 1/6, 5/6 \), we obtain 2766, compared with the true value \( r(601) = 2835 \).
We split the interval \([x_0, x_0 + 1]\) into major arcs, and minor arcs. The major arcs will consist of all those numbers that are within \(x_0\) of a rational number with a denominator not exceeding \((\log n)^{15}\). The minor arcs consist of everything else in \([x_0, x_0 + 1]\); as we shall see later (Theorems 6 and 7) these numbers are always within \(x_0\) of a rational number with a denominator in the range \(((\log n)^{15}, n/(\log n)^{15})\). We attempt to find a reasonably accurate estimate of the integral (1) for \(r(n)\) over the major arcs, where we often expect \(f(x, n)\) to be large. On the minor arcs we are content to find a non-trivial upper bound for \(f(x, n)\).

For a typical major arc, we write

\[
J(h, q) = \int_{h/q + x_0}^{h/q - x_0} f(x, n)^3 e(-nx) \, dx.
\]

We can assume \(n\) is so large that the major arcs do not overlap. Hence

\[
r(n) = \sum_{q \leq (\log n)^{15}} \sum_{0 \leq h \leq q, \gcd(h, q) = 1} J(h, q) + \int_{\text{minor arcs}} f(x, n)^3 e(-nx) \, dx.
\]

The minor arcs

We estimate the integral over the minor arcs with the next few theorems, beginning with an equality which says that adding something small to \(x\) hopefully won’t change \(|f(x, v)|\) too much.

Theorem 4 (151 : Estermann (151)) The function \(f(x, v)\) satisfies the identity

\[
f(x + y, v) = e(vy) f(x, v) - 2\pi iy \int_0^v e(uy) f(x, u) \, du.
\]

Proof Observe that

\[
e(vy) - e(wy) = 2\pi iy \int_w^v e(uy) \, du.
\]

Hence

\[
f(x + y, v) = \sum_{p \leq v} e(px) e(py) = \sum_{p \leq v} e(px) \left( e(vy) - 2\pi iy \int_p^v e(uy) \, du \right)
\]

\[
= e(vy) \sum_{p \leq v} e(px) - 2\pi iy \int_0^v e(uy) \sum_{p \leq u} e(px) \, du
\]

\[
= e(vy) f(x, v) - 2\pi iy \int_0^v e(uy) f(x, u) \, du. \quad \square
\]

Theorem 5 (T56) Let

\[
(\log n)^{15} < q \leq \frac{n}{(\log n)^{15}}, \quad \gcd(h, q) = 1.
\]

and \(v \leq n\). Then

\[
\left| f\left(\frac{h}{q}, v\right) \right| \leq \frac{n}{(\log n)^3}.
\]

Proof Theorem 5 is the vital minor arcs estimate that makes the circle method work for the three primes problem. This is where Vinogradov succeeded after H & L failed. The proof is elementary but complicated and is omitted. See Estermann, pp 54–61. \(\square\)
Theorem 6 (T57) Given any $x$ and any $y \geq 1$, there exist $h$ and $q$ with $q \leq y$ and \(\gcd(h,q) = 1\) such that
\[|qx - h| < \frac{1}{y}.\]

Proof We may assume $0 < x < 1$. Let $m = \lfloor y \rfloor$. Then
\[x \in \left[\frac{h_1}{q_1}, \frac{h_1 + h_2}{q_1 + q_2}\right] \quad \text{or} \quad x \in \left[\frac{h_1 + h_2}{q_1 + q_2}, \frac{h_2}{q_2}\right],\]
where $h_1/q_1$ and $h_2/q_2$ are consecutive fractions in the Farey sequence of order $m$; so that $h_2q_1 - h_1q_2 = 1$ and $q_1 + q_2 \geq m + 1$. If $x$ is in the first interval, we take $h/q = h_1/q_1$ since
\[x - \frac{h_1}{q_1} < \frac{h_2q_1 - h_1q_2}{q_1(q_1 + q_2)} = \frac{1}{q_1(q_1 + q_2)} \leq \frac{1}{q_1(m + 1)} \leq \frac{1}{q_1y}.\]
Similarly we take $h/q = h_2/q_2$ if $x$ is in the second interval. \qed

Theorem 7 (152) Suppose $x$ is in a minor arc. Then
\[|f(x,n)| = O\left(\frac{n}{(\log n)^3}\right)\]

Proof Suppose $x$ is in a minor arc. Then by Theorem 6 with $y = n/(\log n)^{15}$ there exist coprime $h$ and $q$ with
\[(\log n)^{15} < q < \frac{n}{(\log n)^{15}} \quad \text{and} \quad |qx - h| < \frac{(\log n)^{15}}{n} = x_0,\]
the first inequality because otherwise $x$ would be in a major arc. Therefore by Theorem 5
\[\left|f\left(\frac{h}{q}, v\right)\right| \leq \frac{n}{(\log n)^3}, \quad 0 \leq v \leq n.\]
Putting $z = x - h/q$ and using Theorem 4, we have
\[|f(x,n)| = \left|f\left(\frac{h}{q} + z, n\right)\right| = \left|e(nz)f\left(\frac{h}{q}, n\right) - 2\pi iz \int_{0}^{n} e(uz)f\left(\frac{h}{q}, u\right) du\right| \leq \frac{n}{(\log n)^3} + 2\pi z \int_{0}^{n} \frac{n}{(\log n)^3} du \leq \frac{(1 + 2\pi)n}{(\log n)^3}\]
since $|z| = |x - h/q| < x_0/q < 1/n$. \qed

We are now ready to establish the desired non-trivial upper bound for the $r(n)$ integral (1) over the minor arcs.

Theorem 8 (202) We have
\[\int_{\text{minor arcs}} f(x,n)^3 e(-nx)dx = O\left(\frac{n^2}{(\log n)^4}\right).\]
Proof By Theorem 7,
\[ \int_{\text{minor arcs}} f(x,n)^3 e(-nx)dx = O\left( \frac{n}{(\log n)^3} \int_0^1 |f(x,n)|^2 dx \right). \]
But
\[ \int_0^1 |f(x,n)|^2 dx = \sum_{p_1 \leq n} \sum_{p_2 \leq n} \int_0^1 e((p_1 - p_2)x)dx = \sum_{p \leq n} 1 = \pi(n) \]
and the result follows from the prime number theorem (Theorem 1).

\[ \square \]

The major arcs
Let
\[ g(x,v) = \begin{cases} \sum_{2 \leq m \leq v} e(mx) \frac{1}{\log m} & \text{for } v \geq 2, \\ 0 & \text{for } v < 2. \end{cases} \]
So \( g(x,v) \) is like \( f(x,v) \) but instead of summing over primes we sum over integers \( m \) with weight \( 1/(\log m) \) to account for the approximate density of the primes at \( m \).

**Theorem 9 (204)** The function \( g(x,v) \) satisfies the identity
\[ g(x + y, v) = e(vy)g(x,v) - 2\pi iy \int_0^v e(uy)g(x,u)du. \]

Proof Similar to Theorem 4.

\[ \square \]

**Theorem 10 (T58)** Suppose
\[ q \leq (\log n)^{15}, \quad \gcd(h,q) = 1, \quad \text{and } |y| \leq x_0 = (\log n)^{15} n. \]
Then
\[ \left| f\left( \frac{h}{q} + y, n \right) - \frac{\mu(q)}{\phi(q)} g(y,n) \right| \leq \frac{n}{(\log n)^{69}}. \]

Proof Suppose \( v \leq n \). We have
\[ \left| f\left( \frac{h}{q}, v \right) - \sum_{p \leq v, p \nmid q} e\left( \frac{ph}{q} \right) \right| \leq \sum_{p \nmid q} 1 < q. \]

But
\[ \sum_{p \leq v, p \nmid q} e\left( \frac{ph}{q} \right) = \sum_{0 < l \leq q, \gcd(l,q) = 1} e\left( \frac{lh}{q} \right) \sum_{p \leq v \pmod{q}} 1 = \sum_{0 < l \leq q, \gcd(l,q) = 1} e\left( \frac{lh}{q} \right) \pi(v; q, l) \]
and from the prime number theorem for arithmetic progressions (Theorem 2),
\[ \left| \pi(v; q, l) - \frac{\Lambda(v)}{\phi(q)} \right| < \frac{n}{(\log n)^{106}}, \quad \gcd(l, q) = 1. \]
Furthermore, by Lemma 1,

$$\mu(q) = c_q(h) = \sum_{0 < l \leq q, \gcd(l,q)=1} e\left(\frac{lh}{q}\right).$$

Hence, observing that $g(0,v) = ls(v)$,

$$\left| f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0,v)\right| < q + \left| \sum_{p \leq v, p\nmid q} e\left(\frac{ph}{q}\right) - \frac{\mu(q)}{\phi(q)} ls(v)\right|$$

$$= q + \left| \sum_{0 < l \leq q, \gcd(l,q)=1} e\left(\frac{lh}{q}\right) \left(\pi(v;q,l) - \frac{ls(v)}{\phi(q)}\right)\right|$$

$$\leq q + \frac{q n}{(\log n)^{100}} < \frac{2n}{(\log n)^{85}}.$$

Hence by Theorem 4 and Theorem 9,

$$\left| f\left(\frac{h}{q} + y, n\right) - \frac{\mu(q)}{\phi(q)} g(y,n)\right|$$

$$= \left| e(ny) \left(f\left(\frac{h}{q},n\right) - \frac{\mu(q)}{\phi(q)} g(0,n)\right) - 2\pi i y \int_0^n e(vy) \left(f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0,v)\right) dv\right|$$

$$\leq \left| f\left(\frac{h}{q},n\right) - \frac{\mu(q)}{\phi(q)} g(0,n)\right| + 2\pi x_0 \int_0^n \left| f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0,v)\right| dv$$

$$\leq \frac{2n(1 + 2\pi x_0 n)}{(\log n)^{85}} < \frac{14n}{(\log n)^{70}} < \frac{n}{(\log n)^{69}}. \square$$

Observe that for non-square-free $q$, $\mu(q) = 0$ and $f(x,n)$ is small when $x$ is near $h/q$.

**Theorem 11 (T59)** Suppose $q < (\log n)^{15}$, $\gcd(h,q) = 1$ and $|y| \leq x_0$. Then

$$\left| f\left(\frac{h}{q} + y, n\right)^3 - \frac{\mu(q)}{\phi(q)^3} g(y,n)^3\right| \leq \frac{3n^3}{(\log n)^{69}}.$$

**Proof** This follows from Theorem 10 together with the trivial estimates $|f(x,n)| \leq n$ and $|g(x,n)| \leq n$. \square

Substituting $x = h/q + y$ in the expression for $\mathcal{J}(h,q)$ gives

$$\mathcal{J}(h,q) = e\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} f\left(\frac{h}{q} + y, n\right)^3 e(-ny) dy.$$

Putting

$$\mathcal{K} = \int_{-x_0}^{x_0} g(y,n)^3 e(-ny) dy,$$

we have by Theorem 11 and recalling that $x_0 = (\log n)^{15}/n$,

$$\left| \mathcal{J}(h,q) - \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{nh}{q}\right) \mathcal{K}\right| \leq \frac{6n^2}{(\log n)^{54}}, \qquad (2)$$
provided that \( \gcd(h, q) = 1 \) and \( q \leq (\log n)^{15} \).

Just as \( r(n) \) can be expressed as an integral (1) involving \( f(x, n) \), \( \rho(n) \) has a similar formula using \( g(x, n) \):

$$
\rho(n) = \int_{-1/2}^{1/2} g(y, n)^3 e(-ny)dy.
$$

Now for \( w > 2 \) and \( 0 < |y| \leq 1/2 \),

$$
\left| \sum_{m=2}^{w} e(my) \right| = \left| e((w + 1)y) - e(2y) \right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2|y|}.
$$

By Abel’s lemma (summation by parts),

$$
g(y, n) = \frac{1}{\log(n + 1)} \sum_{m=2}^{n} e(my) - \sum_{k=2}^{n} \sum_{m=2}^{k} e(my) \left( \frac{1}{\log(k + 1)} - \frac{1}{\log k} \right),
$$

proved by reversing the order of summation. Hence

$$
|g(y, n)| \leq \frac{1}{2|y| \log(n + 1)} + \frac{1}{2|y|} \sum_{k=2}^{n} \left( \frac{1}{\log k} - \frac{1}{\log(k + 1)} \right) \leq \frac{1}{|y|} \quad \text{for } 0 < |y| \leq 1/2,
$$

and therefore, recalling that \( \mathcal{K} \) is like \( \rho(n) \) but integrating over the shorter interval \([-x_0, x_0]\),

$$
|\rho(n) - \mathcal{K}| \leq \int_{-1/2}^{-x_0} \frac{dy}{y^3} + \int_{x_0}^{1/2} \frac{dy}{y^3} = 2 \int_{x_0}^{1/2} \frac{dy}{y^3} < \frac{1}{x_0^2} = \frac{n^2}{(\log n)^{30}}.
$$

From this and (2) we get a good estimate for a single major arc:

$$
\left| \mathcal{J}(h, q) - \rho(n) \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{nh^5}{q}\right) \right| \leq \frac{6n^2}{(\log n)^{24}} + \frac{n^2}{\phi(q)^3(\log n)^{30}},
$$

again provided that \( \gcd(h, q) = 1 \) and \( q \leq (\log n)^{15} \). Summing over the major arcs and using the definition of Ramanujan’s sum,

$$
\left| \sum_{q \leq (\log n)^{15}} \sum_{0 < h \leq q, \gcd(h, q) = 1} \mathcal{J}(h, q) - \rho(n) \sum_{q \leq (\log n)^{15}} \frac{\mu(q)}{\phi(q)^3} c_q(n) \right| \leq \frac{6n^2}{(\log n)^{24}} + \frac{n^2}{(\log n)^{30}} \sum_{q \leq (\log n)^{15}} \frac{1}{\phi(q)^2} < \frac{7n^2}{(\log n)^{24}} \quad (3)
$$

since \( \sum q \phi(q)^{-2} \) is bounded (recall that \( \phi(n) > 0.3 n^{0.9} \)), and at last we have the estimate over the major arcs that we want.

Combining (3) with the minor arcs estimate (Theorem 8) gives

$$
r(n) - \rho(n) \sum_{q \leq (\log n)^{15}} \frac{\mu(q)}{\phi(q)^3} c_q(n) = O\left(\frac{n^2}{(\log n)^4}\right),
$$

and it is easily shown that the same estimate holds when we take the sum to infinity. Recalling the definition of \( S(n) \) from Theorem 3, we therefore have

$$
r(n) - \rho(n)S(n) = O\left(\frac{n^2}{(\log n)^4}\right),
$$

and the proof of Theorem 3 is complete.