Hadamard and conference matrices

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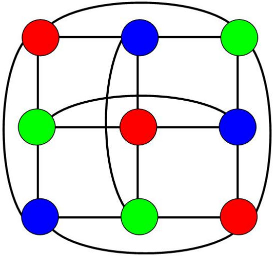
Mathematics Study Group

LONDON SOUTH BANK UNIVERSITY

with input from Rosemary Bailey, Katarzyna Filipiak, Joachim Kunert, Dennis Lin, Augustyn Markiewicz, Will Orrick, Gordon Royle

Happy Birthday, MSG!!



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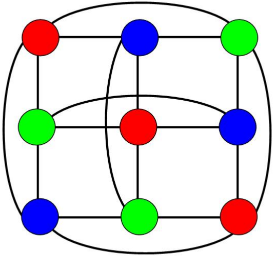

and many happy returns …

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This is a nice example of a continuous problem whose solution brings us into discrete mathematics.

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Examples of Hadamard matrices include

$$(+)\,, \quad \begin{pmatrix} + & + \\ + & - \end{pmatrix}, \quad \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix}.$$

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We can ensure that the first row consists of all $+$s by column sign changes. Then (assuming at least three rows) we can bring the first three rows into the following shape by column permutations:

$$\begin{pmatrix} \overbrace{+ \ldots +}^{a} & \overbrace{+ \ldots +}^{b} & \overbrace{+ \ldots +}^{c} & \overbrace{+ \ldots +}^{d} \\ + \ldots + & + \ldots + & - \ldots - & - \ldots - \\ + \ldots + & - \ldots - & + \ldots + & - \ldots - \end{pmatrix}$$

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Now orthogonality of rows gives

$$a + b = c + d = a + c = b + d = a + d = b + c = n/2,$$

so $a = b = c = d = n/4$.

The Hadamard conjecture

The Hadamard conjecture asserts that a Hadamard matrix exists of every order divisible by 4. The smallest multiple of 4 for which no such matrix is currently known is 668, the value 428 having been settled only in 2005.

Symmetric Hadamard matrices

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In the case where the order is a power of 2, these matrices can be constructed from bent functions (functions on a vector space whose distance from the space of linear functions is maximal). There are connections with coding theory and cryptography.

Skew-Hadamard matrices

A matrix A is skew if $A^\top = -A$.
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It is conjectured that skew-Hadamard matrices of all orders divisible by 4 exist. The smallest unsolved case is 188.

Doubly regular tournaments

If we delete the first row and column of a skew-Hadamard matrix, and replace the diagonal 1s by 0s, we obtain the adjacency matrix of a doubly regular tournament. This means a tournament on $n = 4t + 3$ vertices, in which each vertex has in- and out-degree $2t + 1$, and for any two distinct vertices v and w, there are t vertices z with $v \to z$ and $w \to z$.

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Indeed, Kelly conjectured in the 1960s that every regular tournament has a Hamiltonian decomposition.

An example

$$\begin{pmatrix} 0 & + & + & - & + & - & - \\ - & 0 & + & + & - & + & - \\ - & - & 0 & + & + & - & + \\ + & - & - & 0 & + & + & - \\ - & + & - & - & 0 & + & + \\ + & - & + & - & - & 0 & + \\ + & + & - & + & - & - & 0 \end{pmatrix} \qquad \begin{pmatrix} + & + & + & + & + & + & + & + \\ - & + & + & + & - & + & - & - \\ - & - & + & + & + & - & + & - \\ - & - & - & + & + & + & - & + \\ - & + & - & - & + & + & + & - \\ - & - & + & - & - & + & + & + \\ - & + & - & + & - & - & + & + \\ - & + & + & - & + & - & - & + \end{pmatrix}$$

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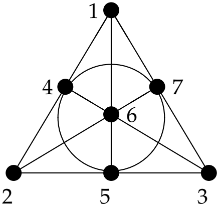
This is related to the Fano plane:



Paley tournaments

The simplest construction of doubly regular tournaments starts with a finite field of order $q \equiv 3 \pmod 4$. The vertices are the elements of the field, and there is an arc $x \to y$ if and only if $y - x$ is a square. (This is a tournament because -1 is a non-square, and therefore $y - x$ is a square if and only if $x - y$ is not.)

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If q is prime, then there is an obvious Hamiltonian decomposition: for each non-zero square s, take the Hamiltonian cycle

$$(0, s, 2s, 3s, \ldots, -s).$$

However, if q is not a prime, it is not so obvious how to proceed.

Conference matrices

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We have:

- The defining equation shows that any two rows of C are orthogonal. The contributions to the inner product of the ith and jth rows coming from the ith and jth positions are zero; each further position contributes $+1$ or -1; there must be equally many (namely $(n-2)/2$) contributions of each sign. So n is even.

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- The defining equation gives $C^{-1} = (1/(n-1))C^\top$, whence $C^\top C = (n-1)I$. So the columns are also pairwise orthogonal.
- The property of being a conference matrix is unchanged under changing the sign of any row or column, or simultaneously applying the same permutation to rows and columns.

Symmetric and skew-symmetric

Using row and column sign changes, we can assume that all entries in the first row and column (apart from their intersection) are $+1$; then any row other than the first has $n/2$ entries $+1$ (including the first entry) and $(n-2)/2$ entries -1. Let C be such a matrix, and let S be the matrix obtained from C by deleting the first row and column.

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Theorem

If $n \equiv 2 \pmod 4$ *then* S *is symmetric; if* $n \equiv 0 \pmod 4$ *then* S *is skew-symmetric.*

Proof of the theorem

Suppose first that S is not symmetric. Without loss of generality, we can assume that $S_{12} = +1$ while $S_{21} = -1$. Each row of S has m entries $+1$ and m entries -1, where $n = 2m + 2$; and the inner product of two rows is -1.

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Suppose that the first two rows look as follows:

$$\begin{array}{cccccc} 0 & + & + \cdots + & + \cdots + & - \cdots - & - \cdots - \\ - & 0 & \underbrace{+ \cdots +}_{a} & \underbrace{- \cdots -}_{b} & \underbrace{+ \cdots +}_{c} & \underbrace{- \cdots -}_{d} \end{array}$$

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From these we obtain

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The other case is similar.

By slight abuse of language, we call a normalised conference matrix C *symmetric* or *skew* according as S is symmetric or skew (that is, according to the congruence on $n \pmod 4$). A "symmetric" conference matrix really is symmetric, while a skew conference matrix becomes skew if we change the sign of the first column.

Symmetric conference matrices

Let C be a symmetric conference matrix. Let A be obtained from S by replacing $+1$ by 0 and -1 by 1.Then A is the incidence matrix of a *strongly regular graph* of Paley type: that is, a graph with $n-1$ vertices in which every vertex has degree $(n-2)/2$, two adjacent vertices have $(n-6)/4$ common neighbours, and two non-adjacent vertices have $(n-2)/4$ common neighbours. The matrix S is called the *Seidel adjacency matrix* of the graph. The complementary graph has the same properties.

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Symmetric conference matrices are associated with other combinatorial objects, among them regular two-graphs, sets of equiangular lines in Euclidean space, switching classes of graphs. A conference matrix can produce many different strongly regular graphs by choosing different rows and columns for the normalisation.

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Again the Paley construction works, on a field of order $q \equiv +1 \pmod 4$; join x to y if $y-x$ is a square. (This time, -1 is a square, so $y-x$ is a square if and only if $x-y$ is.)

An example

The Paley graph on 5 vertices is the 5-cycle. We obtain a symmetric conference matrix by bordering the Seidel adjacency matrix as shown.

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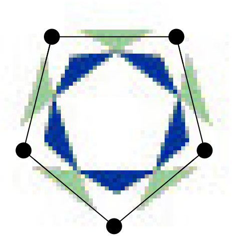

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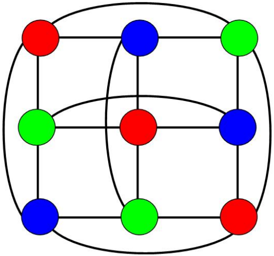

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A new first row and column, with 0 in the $(1,1)$ position and other entries $+$, gives a symmetric conference matrix of order 10.

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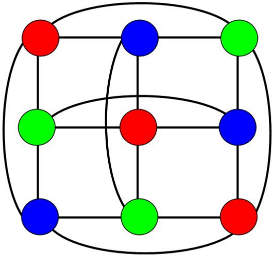

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The MSG logo is the Paley graph on GF(9). (Exercise: Prove this!)

A theorem of van Lint and Seidel asserts that, if a symmetric conference matrix of order n exists, then $n - 1$ is the sum of two squares. Thus there is no such matrix of order 22 or 34. They exist for all other orders up to 42 which are congruent to 2 (mod 4), and a complete classification of these is known up to order 30.

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The simplest construction is that by Paley, in the case where $n-1$ is a prime power: the matrix S has rows and columns indexed by the finite field of order $n-1$, and the (i,j) entry is $+1$ if $j-i$ is a non-zero square in the field, -1 if it is a non-square, and 0 if $i=j$.

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Symmetric conference matrices first arose in the field of conference telephony.

Skew conference matrices

Let C be a "skew conference matrix". By changing the sign of the first column, we can ensure that C really is skew: that is, $C^\top = -C$. Now $(C+I)(C^\top + I) = nI$, so $H = C + I$ is a Hadamard matrix. It is a skew-Hadamard matrix, as defined earlier; apart from the diagonal, it is skew. Conversely, if H is a skew-Hadamard matrix, then $H - I$ is a skew conference matrix.

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If C is a skew conference matrix, then S is the adjacency matrix of a doubly regular tournament, as we saw earlier. (Recall that this is a directed graph on $n-1$ vertices in which every vertex has in-degree and out-degree $(n-2)/2$ and every pair of vertices have $(n-4)/4$ common in-neighbours (and the same number of out-neighbours).

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Again this is equivalent to the existence of a skew conference matrix.

Dennis Lin's problem

Dennis Lin is interested in skew-symmetric matrices C with diagonal entries 0 (as they must be) and off-diagonal entries ± 1, and also in matrices of the form $H = C + I$ with C as described. He is interested in the largest possible determinant of such matrices of given size. Of course, it is natural to use the letters C and H for such matrices, but they are not necessarily conference or Hadamard matrices. So I will call them *cold matrices* and *hot matrices* respectively.

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Of course, if n is a multiple of 4, the maximum determinant for C is realised by a skew conference matrix (if one exists, as is conjectured to be always the case), and the maximum determinant for H is realised by a skew-Hadamard matrix. In other words, the maximum-determinant cold and hot matrices C and H are related by $H = C + I$.

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In view of the skew-Hadamard conjecture, I will not consider multiples of 4 for which a skew conference matrix fails to exist.

A skew-symmetric matrix of odd order has determinant zero; so there is nothing interesting to say in this case. So the remaining case is that in which n is congruent to 2 (mod 4).

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Conjecture

For orders congruent to $2 \pmod 4$*, if* C *is a cold matrix with maximum determinant, then* $C + I$ *is a hot matrix with maximum determinant; and, if* H *is a hot matrix with maximum determinant, then* $H - I$ *is a cold matrix with maximum determinant.*

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Of course, he is also interested in the related questions:

- What is the maximum determinant?
- How do you construct matrices achieving this maximum (or at least coming close)?

Hot matrices

Ehlich and Wojtas (independently) considered the question of the largest possible determinant of a matrix with entries ± 1 when the order is not a multiple of 4. They showed:

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We believe there should be a similar bound for the determinant of a cold matrix.

Meeting the Ehlich–Wojtas bound

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This allows $n = 6, 14, 26$ and 42, but forbids, for example, $n = 10, 18$ and 22.

Computational results

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Lin's conjecture is confirmed for $n = 6$ and $n = 10$. The maximum determinants of hot and cold matrices are $(160, 81)$ for $n = 6$ (the former meeting the EW bound) and $(64000, 33489)$ for $n = 10$ (the EW bound is 73728). In each case there is a unique maximising matrix up to equivalence.

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Random search by Gordon Royle gives strong evidence for the truth of Lin's conjecture for $n = 14, 18, 22$ and 26, and indeed finds only a few equivalence classes of maximising matrices in these cases.

Will Orrick searched larger matrices, assuming a special bi-circulant form for the matrices. He was less convinced of the truth of Lin's conjecture; he conjectures that the maximum determinant of a hot matrix is at least $cn^{n/2}$ for some positive constant c, and found pairs of hot matrices with determinants around $0.45n^{n/2}$ where the determinants of the corresponding cold matrices are ordered the other way.