Tiling Rectangles: 3 Proofs from the BOOK

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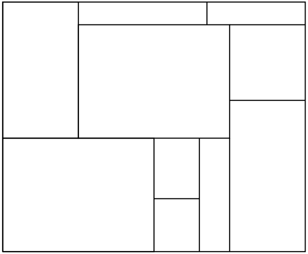
Paul Erdos famously used to say that a particularly beautiful proof was 'one from the BOOK', as though there is a special book of proofs in a 'mathematical heaven', where all the perfect proofs are written.
After he died, Martin Aigner and Gunter Siegler made a collection of beautiful proofs, and called it 'Proofs from the BOOK', from which this was compiled (chapter 26).

Theorem

(Nicholas de Bruijn) If a rectangle can be covered by rectangles which each have at least one integer side, then it also has at least one integer side.



This is at first sight quite a surprising result! Surely if all the tiles all have only integer side, you can have some with the integer side and some vertical, and one might think it possible to construct a rectangle with two non-integer sides from them. But then, it is clear that the theorem is obviously true with just two tiles.

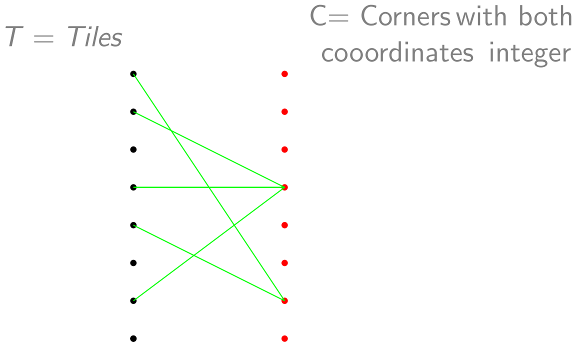
We present three different and ingenious proofs of this fact none of which need induction.

Proof 1, using a graph, by Mike Paterson

Let the bottom left-hand corner of the rectangle be at the origin. Suppose that both the length and width of this rectangle are non-integer, and therefore that all but exactly one of the corners of this rectangle have at least one non-integer coordinate.

Proof 1 continued

We construct a bi-partite graph with vertices T, representing the tiles, and C, representing the corners for which *both* coordinates are integral, and join a member of C to a tile if it meets the tile.



Proof 1 continued

Suppose one corner of a given tile has integer coordinates.

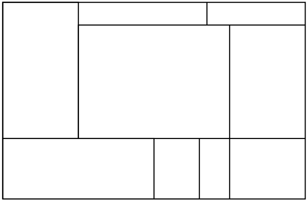
If the length is an integer, then the horizontally adjacent corner also has both integer coordinates.

Alternatively if the height is an integer, the the vertically adjacent corner has both integer coordinates.

Thus in either case at least one other corner of that tile must also have integer coordinates, so every tile has either 2, or 4 corners with integer coordinates.

So the number of edges from the tiles point of view is even.

Proof 1 continued



Now consider the numbers of tiles each member of C can meet. If a given corner c is completely internal, then it is a corner of either 2 or 4 tiles. If the corner c is on an internal part of an edge of the surrounding rectangle, it is a corner of exactly 2 tiles.

Proof 1 continued

Therefore, since by hypothesis the only corner of the rectangle with two integer coordinates is the origin, the number of edges of the graph is odd by this reckoning.

This contradicts the previous reckoning, so our hypothesis must be incorrect.

Therefore, at least one further corner of the rectangle must have integer coordinates. If that is an adjacent vertex, the rectangle has one integer side, if it is the opposite vertex, then both side lengths are integer. QED

Proof 2

This beautiful proof uses 2-D integrals, and is the original proof given by N. de Bruijn

Let the dimensions of the rectangle be LxH, and suppose the coordinates of the lower left and upper right corners of the i^{th} tile are (lx_i, ly_i) and (rx_i, ry_i).

Proof 2 continued

Consider the 2-D integral:

$$\int_c^d \int_a^b e^{i\tau(x+y)} dxdy$$

where for respect with the τ community we are using τ to denote 2π. Now take the above integral over the whole rectangle. But that must be equal to the sum of the same integral over all the individual tiles:

$$\int_0^H \int_0^L e^{i\tau(x+y)} dxdy \sum_{alltiles\ i} \int_{ly_i}^{ry_i} \int_{lx_i}^{rx_i} e^{i\tau(x+y)} dxdy$$

Proof 2 continued

But as is well-known, since we are using Cartesian coordinates, and because of the properties of the exponential function, we can separate the 2-D integral into products:

$$\int_c^d \int_a^v e^{i\tau(x+y)} dxdy = \int_c^d e^{i\tau y} dy \int_a^v e^{i\tau x} dx$$

But taking the 1-D integral:

$$\int_a^b e^{i\tau x} dx = \int_a^b (\cos\tau x + i \sin\tau x) dx = \frac{1}{\tau} [\sin\tau x - i\cos\tau x]_a^b$$

which is zero if $b - a$ an integer, since sin and cos both have period τ.

Proof 2 continued

So, going back to the 2-D integral,

$$\int_c^d \int_a^v e^{i\tau(x+y)} dxdy = \int_c^d e^{i\tau y} dy \int_a^v e^{i\tau x} dx$$

this is zero if and only if either $b - a$ or $d - c$ is an integer.

Proof 2 continued

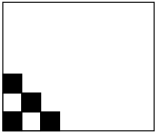
Returning to our tiling,

$$\int_0^H \int_0^L e^{i\tau(x+y)} dxdy \sum_{alltiles\ i} \int_{ly_i}^{ry_i} \int_{lx_i}^{rx_i} e^{i\tau(x+y)} dxdy$$

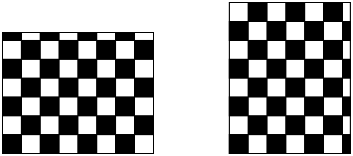
It is clear that since each of the tiles has at least one integer side, all the integrals on the right must be zero, and therefore so is the left-hand integral. But that implies that either L or H is an integer. QED.

Proof 3, due to Rochberg and Stein

This is the simplest and perhaps most surprising of the proofs.
Firstly, we tile whole rectangle with a chequer-board of black and white squares of size $\frac{1}{2}x\frac{1}{2}$, starting with a black square at the origin.



If a tile has an integer side, then the proportions of black to white must be exactly equal, because the number of squares covered is only non-integral in one dimension. So according to the tiling, the coverage is equally black and white.

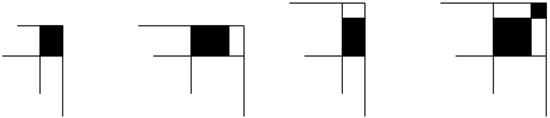


However, if the whole rectangle has no integer side, the proportions cannot be exactly equal.

Proof 3 continued

This seems to be pretty obvious, but to make it explicit, if we split the rectangle into 4 parts by marking off the largest smaller integer width and height, three of the four sub-rectangles has at least one integral dimension, and so have equal proportions of black and white.

However for the final part, there will always be more black than white. There four cases, according to whether the height and width of this final corner rectangle are respective less than or more than $\frac{1}{2}$:



Alternatively, if we let the width and height of the remaining rectagle be and $\frac{1}{2} + y$ respectively, where $-\frac{1}{2} < x, y < \frac{1}{2}$, then the area of black is $\frac{1}{4} + xy$ and the area of white is $\frac{1}{2}x + \frac{1}{2}y$. But:

$$\frac{1}{4} + xy - \frac{1}{2}x - \frac{1}{2}y = (\frac{1}{2} - x)(\frac{1}{2} - y) > 0.$$

So there is strictly more black than white.

So if the dimensions of the rectangle are both non-integer there will always be slightly more more black than white, which contradicts the conclusion from the tiling. Therefore our assumption that the rectangle has no integer sides must be wrong. QED.