Young tableaux and the hook formula

after a paper by Greene, Nijenhuis and Wilf

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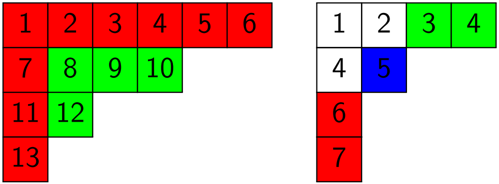
A Young tableau is an array of numbers in rows of non-increasing length, aligned on the left, in which the sequence of numbers in each row and each column is non-decreasing. The tableau is called *standard* if the contents of all the rows and columns are strictly increasing. In this case the entries are usually taken to be $1 \dots n$, where n is the number of entries. The sets of row lengths $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, and column lengths $\lambda^* = (\lambda^*_1, \lambda^*_2, \dots, \lambda^*_k)$, are partitions of n, and are termed the *shape* of the tableau.
Example:

1	2	4	6	8
3	7			
5				
9				

The hook length formula

The hook length formula calculates the number of standard Young tableaux of an given shape. The hook length for the cell in (row,column) (i,j) is the number of cells in the inverted 'L' shape with corner (i,j). Where there is no more more column below or no more row to the right, the the hook is the remainder of the the row, column,or single cell accordingly, viz:



The formula

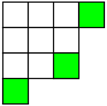
The number of Young tableaux of a given shape λ on $1 \ldots n$ is:

$$F_\lambda = \frac{n!}{\prod_{all \;\; i,j} h_{i,j}} \tag{1}$$

where $h_{i,j}$ stands for the hook length for the cell at coordinate (i,j) of the tableau.
A formula for this was originally discovered as a determinant in 1900/02 by Frobenius and Young, but the formula in this form was derived by Frame, Robinson and Thrall in 1953. Even then proofs which properly involved the hooks took longer to obtain.

Probablistic proof of the hook length formula

In standard Young tableau on $1 \ldots n$, the largest number n must occur on one of the *corners* as shown in green in the diagram:



Moreover, if any one corner is removed, the diagram is again a valid tableau shape. So we can immediately say that F_λ is the sum of the numbers of tableau for shapes with one corner removed.
Therefore we immediately have the identity:

$$F_\lambda = \sum_{\text{all } \alpha} F_{\lambda,\alpha} \tag{2}$$

where $F_{\lambda,\alpha}$ represents the number of standard tableau of the shape made by the removal of the corner at row α.

Since F_λ is 1 for an empty tableau or for one with only one or two cells, this identity can in principle be used to calculate F_λ for any shape. Setting $N_\lambda = \frac{n!}{\prod_{all\ i,j} h_{i,j}}$, we have to prove that $F_\lambda = N_\lambda$. We make the inductive hypothesis that this is true for all tableaux with fewer than n cells. We then prove that

$$\sum_{all\ \alpha} \frac{N_{\lambda,\alpha}}{N_\lambda} = 1 \tag{3}$$

which then from the identity (2) will prove the hook length formula (1). The plan for achieving this is to show that $\frac{N_{\lambda,\alpha}}{N_\lambda}$ can be interpreted as the probability of reaching the corner on row α on the tableau of shape λ by a random walk from any randomly-selected cell. The certainty of reaching some corner is then used to assert equation (3).

We propose a random walk on the tableau of shape λ which starts at a randomly-selected cell (i, j), and from there jumps to a random point on the hook based at (i, j). The walk continues in the same way, moving again to a random point on the hook based at the place it last landed, until a corner is reached, then stops. Since the only allowed movement is horizontally to the right or vertically down, it is inevitable that a corner will eventually be reached. Suppose there is a corner with coordinates (α, β). Let $p(\alpha, \beta)$ be the probability that the random walk will terminate at (α, β).

Theorem

$$p(\alpha, \beta) = \frac{N_{\lambda,\alpha}}{N_\lambda}$$

We first evaluate $\frac{N_{\lambda,\alpha}}{N_\lambda}$. Recall that $N_{\lambda,\alpha}$ relates to the shape obtained when the corner at (α, β) is removed. Removing this corner will have the effect of shortening by one all hooks with their vertical parts in column β, and all hooks with their horizontal parts in row α. Therefore:

$$\begin{aligned} \frac{N_{\lambda,\alpha}}{N_\lambda} &= \frac{1}{n} \frac{\prod_{j<\beta} h_{\alpha,j} \prod_{i<\alpha} h_{i,\beta}}{\prod_{j<\beta} (h_{\alpha,j} - 1) \prod_{i<\alpha} (h_{i,\beta} - 1)} \\ &= \frac{1}{n} \prod_{j<\beta} (1 + \frac{1}{(h_{\alpha,j} - 1)}) \prod_{i<\alpha} (1 + \frac{1}{(h_{i,\beta} - 1)}) \end{aligned} \tag{4}$$

We can then take this expression apart to become a sum of products, involving all numbers of factors of form $\frac{1}{(h_{\alpha,j}-1)}$ and $\frac{1}{(h_{i,\beta}-1)}$. Let I_k for $1 \leq k < \alpha$ be the set of all ordered k-tuples $(1 \leq i_1 < i_2, \ldots < i_k < \alpha)$, and J_k for $1 \leq k < \beta$ be the set of all ordered k-tuples $(1 \leq j_1 < j_2, \ldots < j_k < \beta)$, and additionally I_0 and J_0 as empty sets. Then we can write the right-hand side of (4) as the slightly scary:

$$\frac{1}{n}\sum_{i=0}^{i<\alpha}\sum_{j=0}^{j<\beta}\sum_{A\in I_i}\sum_{B\in J_j}\prod_{s\in A}\prod_{t\in B}\frac{1}{(h_{s,\beta}-1)}\frac{1}{(h_{\alpha,t}-1)}$$

where the empty sets are understood to provide factors of 1.

We assert that each of these products represents the probability of reaching the corner (α, β) by a particular route.
In particular, the value from the empty product is $\frac{1}{n}$ and represents the probability that the walk actually begins at the corner (α, β). Similarly, it is evident that those values with single factors: $\frac{1}{(h_{i,\beta}-1)}$ and $\frac{1}{(h_{\alpha,j}-1)}$ are the respective probabilities that a walk starting at point (i, β) or (α, j) reach the corner at (α, β) in a single move. Likewise products of one type are the probabilities of unidirectional walk downward or rightward respectively.

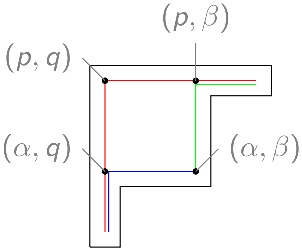
Lemma

If (α, β) is a corner, and $p \leq \alpha$, $q \leq \beta$, then

$$h_{p,q} - 1 = h_{\alpha,q} - 1 + h_{p,\beta} - 1$$

Proof.

This can be inferred from the diagram below (e.g. the corner (α, β) occurs in both $h_{\alpha,q} - 1$ and $h_{p,\beta} - 1$). □



Theorem

if (α, β) is a corner and If $p = i_0 < i_1 \ldots < i_k \leq \alpha)$ and $q = j_0 < j_1 \ldots < j_m \leq \beta$, then the probablity that a random walk on a Young tableau which starts at (p, q) and makes steps across the rectangle defined by all the horizontal coordinates in $i_0, \ldots i_k$ and all the vertical coordinates $j_0, \ldots, j_m$ will reach the corner (α, β) is:

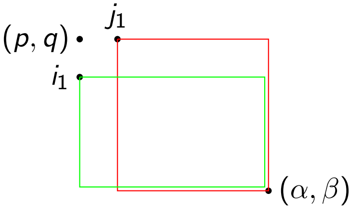
$$\prod_{u=0}^{u=k} \prod_{v=0}^{v=m} \frac{1}{(h_{u,b} - 1)(h_{a,v} - 1)}$$

Proof of theorem

We prove this by induction. With the convention that an empty product has value 1, the theorem holds for $k = m = 0$. Also if $p = \alpha$ or $q = \beta$, then the probability is clearly

$$\prod_{v=0}^{v=m} \frac{1}{(h_{\alpha,v} - 1)} \; or \; \prod_{u=0}^{u=k} \frac{1}{(h_{u,\beta} - 1)}$$

respectively. So we assume that the theorem is true for all all rectangles smaller then $k \times m$. Now consider the two rectangles bounded by $(p, i_1, \ldots, i_k, \alpha) \times (j_1, \ldots, j_m, \beta)$ and $(i_1, \ldots, i_k, \alpha) \times (q, j_1, \ldots, j_m, \beta)$ respectively as in the figure below:



It is a standard result in statistics that if an event Z can result from either of two exclusive events X and Y, then

$$p(Z) = p(Z|X)p(X) + p(Z|Y)p(Y)$$

So the the probability we want is

$$pr((i_1, q)|(p, q))pr((\alpha, \beta)|(i_1, q)) + pr((p, j_1)|(p, q))pr(\alpha, \beta)|(p, j_1))$$

$$= \frac{1}{(h_{p,q}-1)} \prod_{u=1}^{u=k} \prod_{v=0}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

$$+ \frac{1}{(h_{p,q}-1)} \prod_{u=0}^{u=k} \prod_{v=1}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

$$= \frac{1}{(h_{p,q}-1)} \left(\frac{1}{(h_{p,\beta}-1)} + \frac{1}{(h_{\alpha,q}-1)} \right) \prod_{u=1}^{u=k} \prod_{v=1}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

$$= \frac{1}{(h_{p,q}-1)} \left(\frac{h_{p,\beta}-1+h_{\alpha,q}-1}{(h_{p,\beta}-1)(h_{\alpha,q}-1)} \right) \prod_{u=1}^{u=k} \prod_{v=1}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

$$= \frac{1}{(h_{p,\beta}-1)(h_{\alpha,q}-1)}) \prod_{u=1}^{u=k} \prod_{v=1}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

by the Lemma.

$$= \prod_{u=0}^{u=k} \prod_{v=0}^{v=m} \frac{1}{(h_{u,b}-1)} \frac{1}{(h_{a,v}-1)}$$

which proves the theorem.

Returning to our proof of the hook length formula, we can see that in the expression

$$\frac{1}{n}\sum_{i=0}^{i<\alpha}\sum_{j=0}^{j<\beta}\sum_{A\in I_i}\sum_{B\in J_j}\prod_{s\in A}\prod_{t\in B}\frac{1}{(h_{s,\beta}-1)}\frac{1}{(h_{\alpha,t}-1)}$$

each set of products represents the probability of reaching the corner (α, β) across a rectangle defined by particular sets of horizontal and vertical coordinates A and B. Each rectangle is an exclusive event, and the sum is over all possible rectangles with all possible starting corners. The factor $\frac{1}{n}$ represents the the choice of starting corner. Hence the total expression for $\frac{N_\alpha}{N}$ represents the probability of a random walk starting from a random place reaching the corner (α, β).

Since we are certain that any random walk must reach some corner, we can conclude that

$$\sum_{allcorners(\alpha,\beta)} \frac{N_\alpha}{N_\lambda} = 1$$

and so by the inductive hypothesis

$$F_\lambda = N_\lambda = \frac{n!}{\prod_{all \ i,j} h_{i,j}}$$

as required.