Partial Fields and Matroid Representation

What is a matroid?

Matroids capture the combinatorial properties of a finite set of vectors. They play a role in discrete mathematics analogous to that played by topology in continuous mathematics or group theory in algebra.

Theme of Talk

In essence, matroid theory is a branch of modern projective geometry.

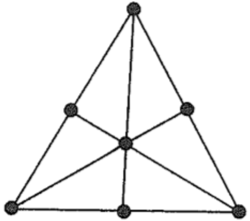
Canonical Example

$\mathbb{F}$ a field; S a set of vectors over $\mathbb{F}$. We have a matroid whose independent sets are the subsets of S that are linearly independent over $\mathbb{F}$.

Say $\mathbb{F} = \mathbb{R}$. Let $S = \{a, b, c, d, e, f, g\}$. Then

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

defines a matroid M on S.



F_7^-

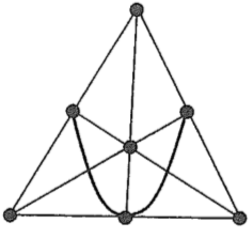
- A matroid is representable over $\mathbb{F}$ if it can be obtained from a set of vectors over $\mathbb{F}$.
- The vectors in S can be compactly described as the columns of a matrix A. In this case M is the column matroid of A.
- A set of j columns in a matrix is independent if and only if it contains a $j \times j$ submatrix whose determinant in nonzero.
- Row operations do not affect linear independence of columns, therefore they do not change the matroid.

Different Fields, Different Matroids

Recall the matrix

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

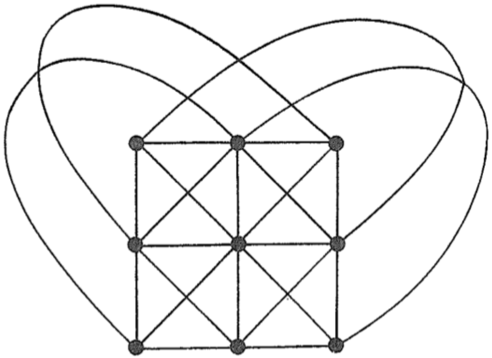
But change the field to $\mathrm{GF}(2)$.

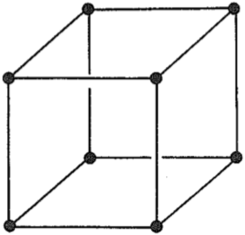


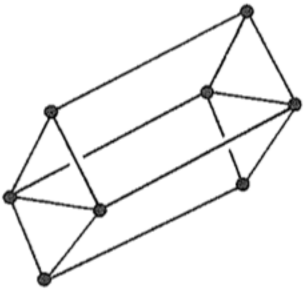
F_7

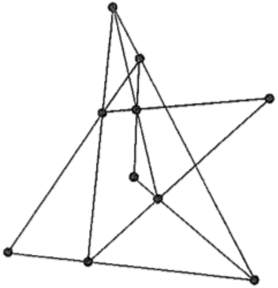
Which matroids are representable over which fields?

- F_7^- is representable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is not equal to 2.
- F_7 is representable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is equal to 2.









Regular Matroids

A matrix over $\mathbb{R}$ is unimodular if every square submatrix has a determinant in $\{0, 1, -1\}$.

A matroid is regular if every it can be represented by a unimodular matrix.

Theorem (Tutte 1954)

The following are equivalent.

- *M is regular.*
- *M is representable over every field.*
- *M is representable over $GF(2)$ and $GF(3)$.*
- *M is representable over $GF(2)$ and $\mathbb{F}$ where $\mathbb{F}$ is any field whose characteristic is not 2.*

Theorem

Let $\mathcal{F}$ be a set of fields containing $GF(2)$ and $\mathcal{M}$ be the set of matroids representable over all fields in $\mathcal{F}$. Then $\mathcal{M}$ is either the class of regular matroids, or the class of binary matroids.

Only two classes arise.

Ternary Matroids

M is ternary if it representable over GF(3). What classes arise there?

Dyadic Matroids

A matrix over $\mathbb{R}$ is dyadic if all nonzero subdeterminants are in $\{\pm 2^i : i \in \mathbb{Z}\}$.
A matroid is dyadic if it can be represented by a dyadic matrix.

Theorem

The following are equivalent.

- *M is dyadic.*
- *M is representable over $GF(3)$ and $GF(5)$.*
- *M is representable over $GF(3)$ and $\mathbb{Q}$.*
- *M is representable over $GF(3)$ and $\mathbb{R}$.*

What about GF(3) and $\mathbb{C}$?

Sixth-root of unity matroids

A matrix over $\mathbb{C}$ is *sixth-root of unity* if every nonzero subdeterminant is a sixth-root of unity.

A sixth-root of unity matroid is one that can be represented by a sixth-root of unity matrix.

Theorem

The following are equivalent.

- *M is a sixth-root of unity matroid.*
- *M is representable over $GF(3)$ and $GF(4)$.*

What about GF(3) and $\mathbb{C}$?

Near-regular matroids

A matrix over $\mathbb{Q}(\alpha)$ is near-regular if all nonzero subdeterminants are in $\{\alpha^i(\alpha - 1)^j : i, j \in \mathbb{Z}\}$.
A matroid is near-regular if it can be represented by a near-regular matrix.

Theorem

The following are equivalent.

- *M is near-regular.*
- *M is representable over all fields other than possibly $GF(2)$.*
- *M is representable over $GF(3)$ and $GF(8)$.*

Theorem

Let $\mathcal{F}$ be a set of fields containing $GF(3)$. Then there is an $i \in \{2, 3, 4, 5, 7, 8\}$ such that the class of matroids representable over all fields in $\mathcal{F}$ is the class of matroids representable over $GF(3)$ and $GF(i)$.

What is a partial field?

- Essentially a partial field is a set $\mathbb{P}$ containing $\{0, 1\}$ with a multiplicative group and a partial addition.
- Can develop a theory of matroid representation over partial fields.
- Canonical examples. Let $\mathbb{F}$ be a field and let G be a subgroup of $\mathbb{F}^*$ containing -1. Then $G \cup \{0\}$ with the induced operations is a partial field.
- Can also define a partial field by generators and relations in a natural way.

Homomorphisms

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be partial fields. Then a function $\phi : \mathbb{P}_1 \to \mathbb{P}_2$ is a homomorphism if blah blah blah and, whenever $x + y$ is defined, then $\phi(x) + \phi(y)$ is defined.

Lemma

If there is a non-trivial homomorphism from $\mathbb{P}_1$ to $\mathbb{P}_2$, then every matroid representable over $\mathbb{P}_1$ is also representable over $\mathbb{P}_2$.

Standard Constructions

Matroids representable over a partial field are closed under standard matroid operations, ie duality, direct sums, 2-sums, minors etc.

This reduces many combinatorial/geometric arguments to routine algebra.

Theorem (Vertigan)

Every partial field can be obtained by restricting to a subgroup of the group of units of a commutative ring.

Theorem (Vertigan)

Let $\mathcal{F}$ be a set of fields.

- *The matroids representable over all fields in $\mathcal{F}$ is the class of matroids representable over a partial field.*
- *The matroids representable over at least one field in $\mathcal{F}$ is the class of matroids representable over a partial field.*

Theorem (Vertigan)

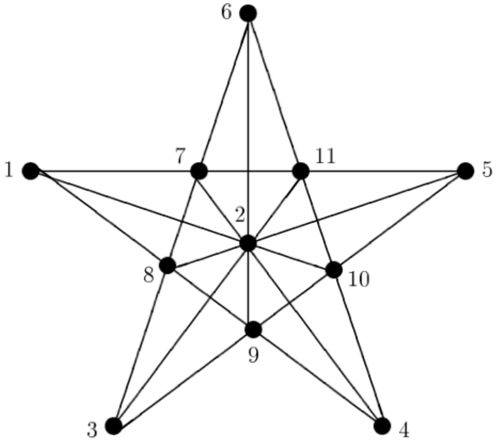
If M is representable over some partial field. Then there exists a field over which M is representable.

Golden-ratio matroids

Let r and $1-r$ be the roots of x^2-x-1 over $\mathbb{R}$, and let $\mathbb{GM}$ denote the set $\{r^i(1-r)^j : i,j \in \mathbb{Z}\}$ with the induced operations from $\mathbb{R}$. Then $\mathbb{GM}$ is the golden-ratio partial field. Matroids representable over $\mathbb{GM}$ are golden-ratio matroids.

Theorem (Vertigan)

A matroid is representable over $GF(4)$ and $GF(5)$ if and only if it is a golden-ratio matroid.


6
7
11
1
5
2
8
10
9
3
4

:-(

And then the subject died.

:-)

Until Stefan van Zwam.

Associates and fundamental elements

An element a of a partial field is fundamental if $a - 1$ is defined. If a is fundamental, then all members of

$$\left\{a, 1 - a, \frac{1}{1 - a}, \frac{a}{a - 1}, \frac{a - 1}{a}, \frac{1}{a}\right\}$$

are fundamental. The members of the above set are the associates of a.

Representations of 4-point lines

Consider a 4-point line represented by

$$\begin{array}{cccc} d & e & f & g \end{array}$$
$$\begin{pmatrix} d_1 & e_1 & f_1 & g_1 \\ d_2 & e_2 & f_2 & g_2 \end{pmatrix}$$

Using row operations and column scaling, this is equivalent to

$$\begin{array}{cccc} d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{pmatrix}$$

Then x is the cross ratio $de : fg$ (or something like it!) Note that x is fundamental.

The associates of x are precisely the set of values you get for other cross ratios involving d, e, f, and g.

Nothing new under the sun

- Fundamental elements are allowable cross ratios in "4-point line" minors of $\mathbb{P}$-represented matroids.
- Harmonic and Equienharmonic cross ratios.

Restatement of Vertigan's Theorem

A matroid is representable over $GF(4)$ and $GF(5)$ if and only if it has a representation over $\mathbb{R}$ with the property that the cross ratios of every induced representation of every 4-point line minor are golden ratios - or associates thereof.

Quaternary matroids

What about matroids representable over fields containing $GF(4)$?

- The 2-regular partial field naturally generalises near regular; 2-regular matroids are representable over all fields of size at least 4.
- Class of matroids representable over all fields of size at least 4 strictly contains 2-regular matroids.
- There are an infinite number of classes that arise when we consider matroids representable over $GF(4)$ and other fields.

Overall feeling

It seems like we've hit a bit of a wall. Has the algebraic bus has reached its terminus and are we back to grungy geometric/combinatorial/connectivity arguments?

I don't really believe it.

let $\mathcal{R}(q)$ denote the set of matroids representable over all fields with at least q elements.

Theorem

There are infinitely many Mersenne primes if and only if, for each prime power q, there is an integer m_q such that a 3-connected member of $\mathcal{R}(q)$ has at most m_q inequivalent $GF(7)$-representations.