Regular polytopes

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Old Codgers Combinatorics Colloquium
Reading, 5 November 2014

Acknowledgement

This is joint work with Maria Elisa Fernandes (Aviero), Dimitri Leemans (Auckland) and Mark Mixer (Boston).
This year I spent nearly two months in Auckland, thanks to support from the Hood Fellowship, where I learned about regular polytopes and carried out the research reported here.

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I begin with something that seems at first glance to have nothing at all to do with polytopes, but there is a connection …

Independent generating sets

Let G be a finite group. A set $\{g_1, \ldots, g_r\}$ of elemets of G is independent if none of the elements lies in the subgroup generated by the others. It is an independent generating set if, in addition, the whole set generates the group G.

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Thus independent generating sets resemble bases for vector spaces in elementary linear algebra. However, they do not have the nice properties of bases such as the exchange property, and so they are not the bases of a matroid.

In the symmetric group

Theorem (Julius Whiston, 2000)

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There are a few extra types for small n. For example, for $n = 6$, we can take images of the above types under the outer automorphism of S_6.

Subgroup lattices

Let $L(G)$ denote the subgroup lattice of the group G.

Proposition

For any finite group G, the Boolean lattice $B(r)$ is embeddable as a meet-semilattice of $L(G)$ if and only if it is embeddable as a join-semilattice of $L(G)$. The largest number r for which these equivalent properties hold is equal to the size of the largest independent subset of G.

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If $\{g_1, \ldots, g_r\}$ is an independent set in G, then the subgroups generated by its subsets form a join-semilattice of $L(G)$ isomorphic to $B(r)$.
Note that the above conditions are not equivalent to the embeddability of $B(r)$ in $L(G)$ as a lattice!

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We require several further conditions (see next slide).

- For $i < j < k$, if x, y, z are elements of dimensions i, j, k with $x \leq y$ and $y \leq z$, then $x \leq z$.

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- A strong connectedness condition: if F and G are two flags, then there is a sequence of flags beginning at F and ending at G, such that consecutive members intersect in all but one of their elements, and that $F \cap G$ is contained in every flag in the sequence.

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If x and y are elements of a polytope with $x < y$, then the interval $[x, y] = \{z : x \leq z \leq y\}$ is itself a polytope, of dimension $\dim(y) - \dim(x) - 2$. In particular, if $\dim(y) - \dim(x) = 3$, then $[x, y]$ is a polygon.

Regular polytopes

If two flags $(x_{-1}, x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r)$ and $(x_{-1}, x_0, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_r)$ differ only in the element of dimension i, then any automorphism which fixes the first flag also fixes the second.

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If a polytope is regular, then for any i, if $\dim(x) = i - 1$, $\dim(y) = i + 2$, and $x < y$, then $[x, y]$ is a p_i-gon, where p_i depends on i but not on x and y. The vector $(p_0, p_1, \ldots, p_{r-1})$ is the Schläfli symbol of the polytope.

String C-groups

Because of the correspondence between the set of flags and the automorphism group G of a polytope, it is possible to translate everything into the group. We will see that the existence of a regular polytope is equivalent to a sequence of group elements with certain properties.

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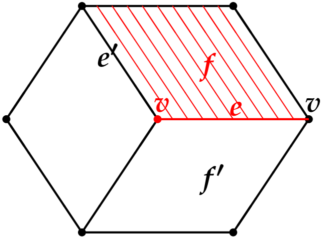
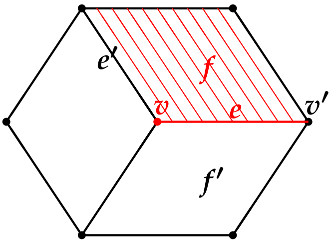
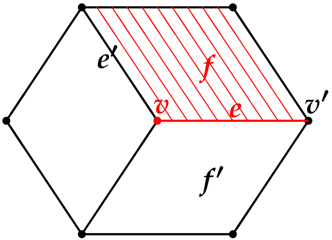
To motivate this, consider the cube.



Our reference flag is $(\emptyset, v, e, f, C)$ (where C denotes the cube).
Let s_v, s_e and s_f be the automorphisms mapping it to $(\emptyset, v', e, f, C)$, $(\emptyset, v, e', f, C)$ and $(\emptyset, v, e, f', C)$ respectively.



Now s_v maps v' back to v, and so $s_v^2 = 1$; similarly $s_e^2 = s_f^2 = 1$.



Now s_v maps v' back to v, and so $s_v^2 = 1$; similarly $s_e^2 = s_f^2 = 1$. Also $s_v s_e$ rotates the square face f one step clockwise, and so $(s_v s_e)^4 = 1$. Similarly $(s_e s_f)^3 = 1$. And s_v and s_f both fix e, and so they commute: $(s_v s_f)^2 = 1$.

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Theorem

The existence of a regular polytope with automorphism group G is "equivalent" (in a suitable sense) to a representation of G as a string C-group.

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We do not insist that s_i and s_j fail to commute if $|i - j| > 1$. In other words, we allow *degenerate* polytopes where some of the polygons are digons. This might seem to make things harder, but actually makes them much easier. The subgroup generated by a subset of $\{s_0, \ldots, s_{r-1}\}$ is a string C-group in its own right, so we have the possibility of induction!

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Also, we do not assume that the orders of the s_i and $s_i s_j$ give a *presentation* of a group. (If they do, then the group is a *Coxeter group*.)
Finally, the intersection condition shows that $\{s_0, \ldots, s_{r-1}\}$ is an independent generating set for G. Indeed, it is stronger: it is equivalent to the condition that the map $I \mapsto G_I$ embeds the Boolean lattice $B(r)$ as a sublattice of the subgroup lattice $L(G)$ of G.

The symmetric group, 1

It follows from Whiston's theorem that the dimension of a polytope with autmorphism group S_n is at most $n-1$. It further follows from the theorem of Cameron and Cara that there is a unique such polytope of rank $n-1$. (The condition that generators are involutions rules out the second type; the string condition shows that the tree is a string.) The generators are $s_i = (i+1, i+2)$ for $i = 0, \ldots, n-2$.

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The corresponding polytope is the regular $(n-1)$-simplex, whose faces are all the subsets of $\{1, \ldots, n\}$.

The symmetric group, 2

Fernandes, Leemans and Mixer asked about regular polytopes of smaller dimension r with group S_n. They computed the following table:

$n \backslash r$	3	4	5	6	7	8	9	10	11	12	13
5	4	1	0	0	0	0	0	0	0	0	0
6	2	4	1	0	0	0	0	0	0	0	0
7	35	7	1	1	0	0	0	0	0	0	0
8	68	36	11	1	1	0	0	0	0	0	0
9	129	37	7	7	1	1	0	0	0	0	0
10	413	203	52	13	7	1	1	0	0	0	0
11	1221	189	43	25	9	7	1	1	0	0	0
12	3346	940	183	75	40	9	7	1	1	0	0
13	7163	863	171	123	41	35	9	7	1	1	0
14	23126	3945	978	303	163	54	35	9	7	1	1

We see the entries 1 for $r = n - 1$ corresponding to the regular simplices, and we have seen that there are no more. Note also the entries 1 for $r = n - 2$, $n \geq 7$; 7 for $r = n - 3$, $n \geq 9$; 9 for $r = n - 4$, $n \geq 11$; and 35 for $r = n - 5$, $n \geq 13$.

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This suggests the conjecture:

Conjecture

Given k, there is a number $N(k)$ such that, for $n \geq 2k + 3$, the number of regular polytopes of dimension $n - k$ with automorphism group S_n is $N(k)$.

Fernandes, Leemans and Mixer have established this conjecture for $k \leq 4$, with the values of $N(k)$ given above.

The alternating groups

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This, incidentally, shows that there is a big difference between largest dimension of a polytope with group G, and largest independent generating set for G (which is $n-2$ for $G = A_n$).

Other subgroups of S_n

My contribution to this problem, after working intermittently with Dimitri Leemans on this, was the following theorem. ("Number" refers to the list of transitive groups of this degree in Magma.)

Theorem

A regular polytope of rank r whose group G is isomorphic to a transitive subgroup of S_n other than S_n or A_n satisfies one of the following:

- $r \leq n/2$.
- $n \equiv 2 \mod 4$, $r = n/2 + 1$ *and G is* $C_2 \wr S_{n/2}$. *The generators are explicitly known; the Schäfli type is* $(2, 3, \ldots, 3, 4)$.
- *G is transitive imprimitive and is one of the examples appearing in the table below.*
- *G is primitive. In this case, G is obtained from the permutation representation of degree 6 of* $S_5 \cong PGL_2(5)$ *and the polytope is the 4-simplex of Schäfli type* $[3, 3, 3]$.

Degree	Number	Structure	Order	Schäfli type
6	9	$S_3 \times S_3$	36	$[2,3,3]$
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We hope to be able to use this result to prove the conjectured bound for the dimension of a polytope admitting the alternating group. It may also be of use in tackling the mysterious conjecture for the symmetric group.