Transforms, minors and generalised Tutte polynomials

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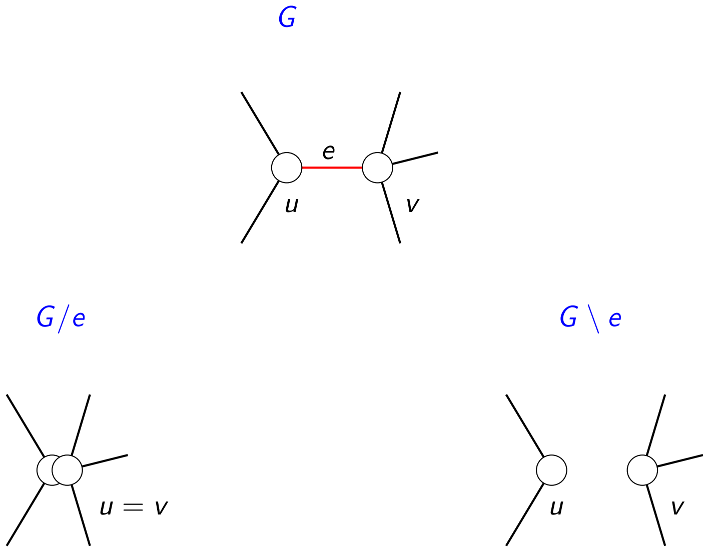
(work done partly at the Isaac Newton Institute for Mathematical Sciences (Combinatorics and Statistical Mechanics Programme), Cambridge, 2008)

14 January 2010

Contraction and Deletion



Cutset space

Incidence matrix of graph G:

$$\text{vertices} \quad \overset{\text{edges}}{\begin{pmatrix} & \vdots & \\ \cdots & 0/1 \text{ entries} & \cdots \\ & \vdots & \end{pmatrix}}$$

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Cutset space := rowspace of incidence matrix over $GF(2)$.

Indicator function of cutset space:

$f : 2^E \to \{0,1\}$, defined by:

$$f(X) = \begin{cases} 1, & \text{if } X \text{ is in cutset space;} \\ 0, & \text{otherwise.} \end{cases}$$

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$f : 2^E \rightarrow \{0, 1\}$, defined by:

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Contraction and deletion in terms of f

Indicator function of cutset space of G:

$$f : 2^E \to \{0, 1\}$$

For contraction and deletion of some $e \in E$:
Indicator functions of cutset spaces of ...

G/e

$$f /\!\!/ e : 2^{E \setminus \{e\}} \to \{0, 1\}$$

$$f /\!\!/ e\,(X) = \frac{f(X)}{f(\emptyset)}$$

$G \setminus e$

$$f \setminus\!\!\setminus e : 2^{E \setminus \{e\}} \to \{0, 1\}$$

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For any $f : 2^E \to \{0,1\}$ ... or ... $\to \mathbb{R}$ ... :
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For $X \subseteq E(G)$,
$rank(\mathrm{X}) :=$ #vertices meeting X − #components of X
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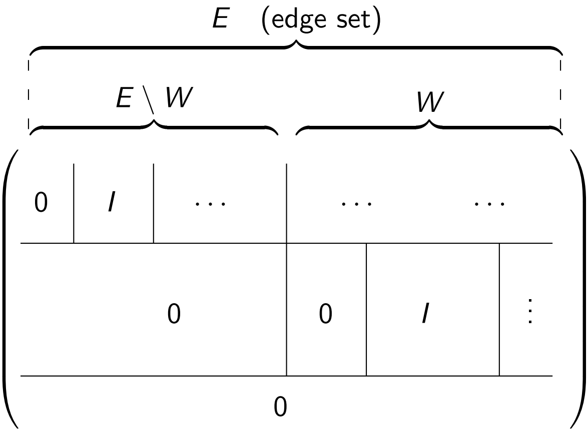

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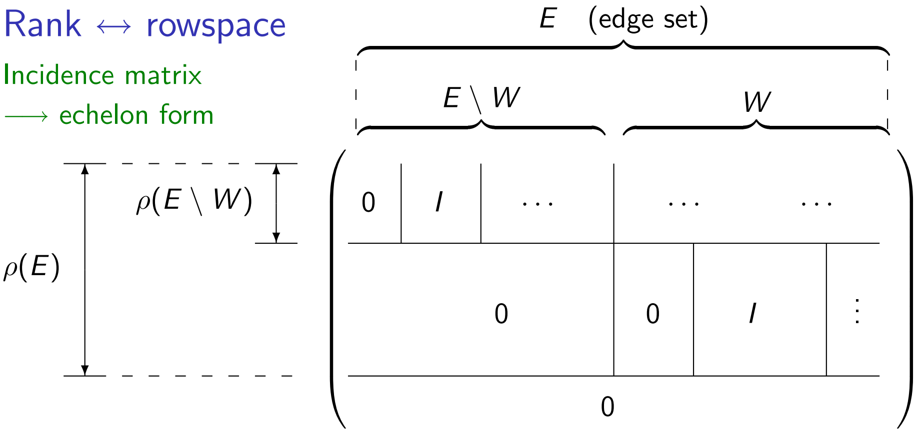

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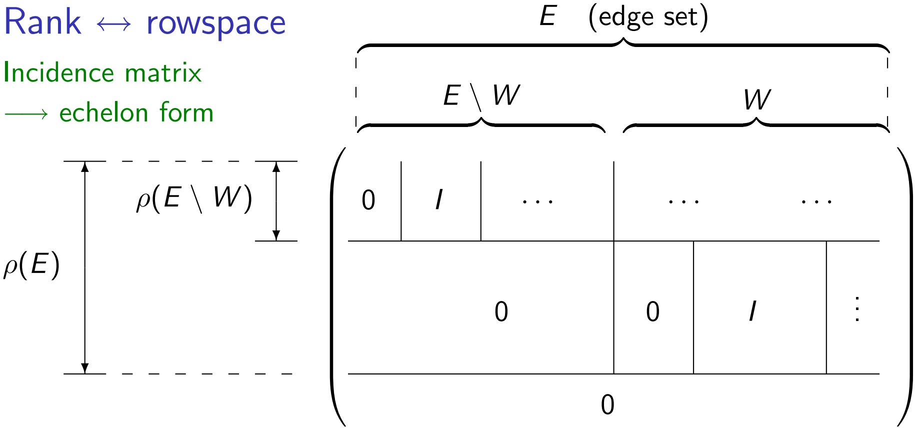

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$$2^{\rho(E)-\rho(E\setminus W)} = \sum_{X \subseteq W} \mathrm{ind}_{\mathrm{Rowspace}}(X)$$

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Duality and the Hadamard transform

Classical rank function duality:

$$\rho^*(X) := |X| + \rho(E \setminus X) - \rho(E) + \rho(\emptyset)$$

Hadamard transform:

$$\hat{f}(W) := \frac{1}{2^n} \sum_{X \subseteq E} (-1)^{|W \cap X|} f(X)$$

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These are linked by the rank transform Q (GF, 1993):

$$\begin{array}{rcl} f & \xrightarrow{\quad Q \quad} & Qf \\ \text{Hadamard transform} \downarrow & & \downarrow \text{matroid-style dual} \\ \hat{f} & \xrightarrow{\quad Q \quad} & (Qf)^* = Q\hat{f} \end{array}$$

Interpolating between contraction and deletion

(GF, 2004)
For $e \in E$, $X \subseteq E \setminus \{e\}$:

Contraction	Deletion
$(f \mathbin{/\!\!/} e)(X)$	$(f \mathbin{\backslash\!\!\backslash} e)(X)$
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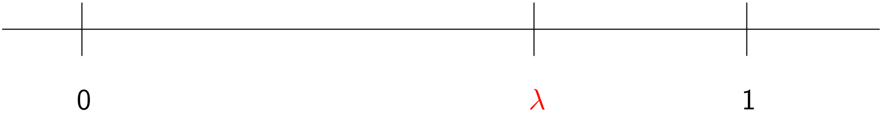

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Fixed points:

$$\lambda = \pm\sqrt{2} - 1$$

λ-rank functions

Define $Q^{(\lambda)}f$ by:

$$(Q^{(\lambda)}f)(W) = \log_2 \left(\frac{(1+\lambda^*)^{|V|} \sum_{X \subseteq E} \lambda^{|X|} f(X)}{\sum_{X \subseteq E \setminus W} \lambda^{|W \cap \bar{V}|} (\lambda^*)^{|W \cap V|} f(X)} \right)$$

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Duality:

$$(Q^{(\lambda)})^* = Q^{(\lambda^*)}$$

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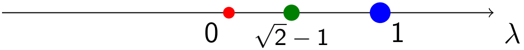
Leads to a theory of λ-Whitney functions, extending the Whitney rank generating function of a graph or matroid (GF, 2004, 2007).

From λ to μ



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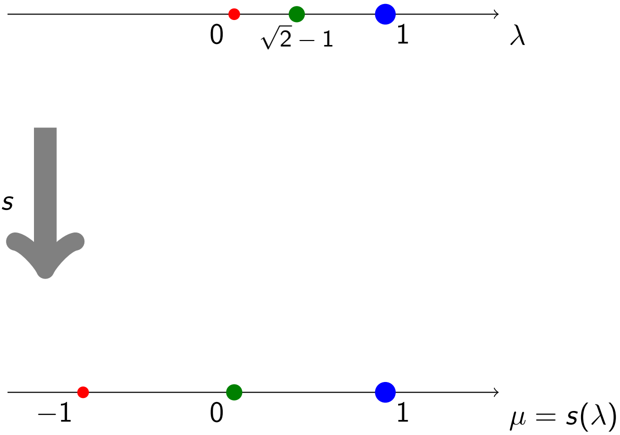

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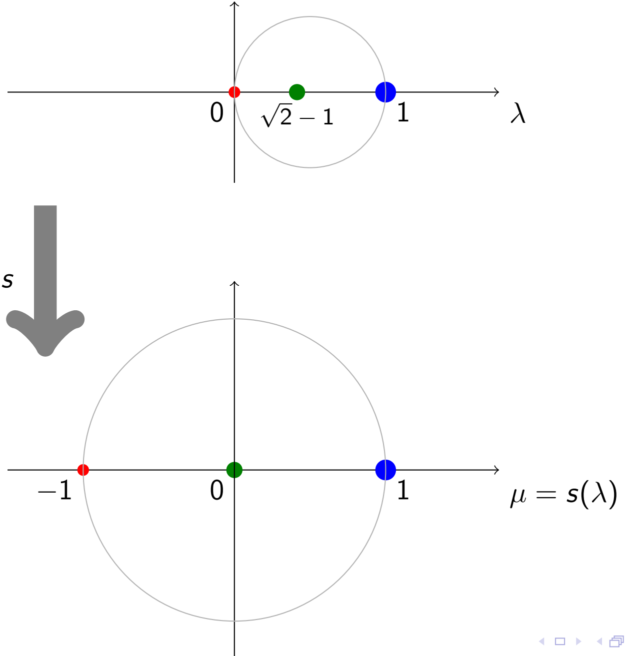

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From λ to μ

$$\mu = s(\lambda) \quad := \quad -(3+2\sqrt{2})\,\frac{\sqrt{2}-1-\lambda}{\sqrt{2}+1+\lambda}$$

$$\lambda = s^{-1}(\mu) \quad := \quad \frac{1+\mu}{\sqrt{2}+1-(\sqrt{2}-1)\mu}$$

Notation:

$$Q^{[\mu]} \quad := \quad Q^{(s^{-1}(\mu))}$$

$$f \,\|_{[\mu]}\, e \quad := \quad f \,\|_{s^{-1}(\mu)}\, e$$

The transform $L^{[\mu]}$

$$(L^{[\mu]}f)(V) = (2\sqrt{2})^{-|E|} \times \sum_{X \subseteq E} (\sqrt{2}-1+(\sqrt{2}+1)\mu)^{|X \cap V|} \cdot (1-\mu)^{|X \setminus V|+|V \setminus X|} \cdot (\sqrt{2}+1+(\sqrt{2}-1)\mu)^{|E \setminus (X \cup V)|} \; f(X)$$

Special cases:

$\mu = 1$: identity transform

$\mu = -1$: $\sqrt{2}^{|E|} \times$ Hadamard transform

Properties of the transforms

Composition of transforms $\longleftrightarrow$ multiplication of their parameters:

$$L^{[\mu_1]}L^{[\mu_2]} = L^{[\mu_1\mu_2]}$$

$$Q^{[\mu_1]}L^{[\mu_2]} = Q^{[\mu_1\mu_2]}$$

$$Q^{\dagger[\mu_1]}Q^{[\mu_2]}f = \text{constant} \cdot A_f(s^{-1}(\mu_2))^{-1} \cdot L^{[\mu_2/\mu_1]}f$$

... where $A_f(z) := \sum_{X \subseteq E} f(X) z^{|X|}$
(weight enumerator of $\operatorname{supp} f$, if f is $\{0,1\}$-valued)

$$Q^{[\mu_1]}Q^{\dagger[\mu_2]}Q^{[\mu_2]} = Q^{[\mu_1\mu_3/\mu_2]}$$

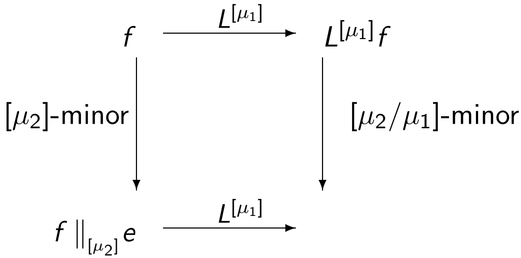
Also have generalisations of Plancherel's and Parseval's theorems.

$[\mu]$-minors

Theorem

$$(L^{[\mu_1]} f) \|_{[\mu_2/\mu_1]} e = \mathit{ScalingFactor}(f, \mu_1, \mu_2) \cdot L^{[\mu_1]}(f \|_{[\mu_2]} e)$$

Up to constant factors:



An extension of MacWilliams' Identity

Theorem

$$A_{L^{[\mu_2/\mu_1]}f}(s^{-1}(\mu_1)^*) = \mathit{factor}(\mu_1, \mu_2, m) \cdot A_f(s^{-1}(\mu_2)^*).$$

For original MacWilliams identity: put

$$\begin{aligned} f &= \text{indicator function of binary linear code,} \\ \mu_2 &= -\mu_1 \end{aligned}$$

The $L^{[\mu]}$ transform and the weight enumerator

Theorem

$$L^{[\mu]}\left(2^{|E|}\cdot a_1(\mu)^{|\bullet|}\cdot \hat{f}(\bullet)\right)(E) = a_2(\mu)^{|E|}A_f\left(\left(\frac{1-\mu}{1+\mu}\right)^2 - 1\right).$$

If f is the indicator function of a binary linear code, and $\mu \in \mathbb{C}$ with $|\mu| = 1$, then this result connects ...

$L^{[\mu]}$ transform (with μ on complex unit circle) of close relative of indicator function of *dual* code	$\longleftarrow \cdots \longrightarrow$	weight enumerator of binary linear code at real argument < -1