**THEOREM OF THE DAY**

**Babbitt’s Theorem** Suppose that $S$ is a subset of the set of pitch classes comprising $\mathbb{Z}_n$, with interval class vector $v$. Let $S'$ be the transposition of $S$ by an interval $k \leq n/2$, that is, $S' = S + k$, with addition being modulo $n$. Then the number of pitches common to both $S$ and $S'$ is equal to the $k$-th entry of $v$: $|S \cap S'| = v_k$.

A little amateur musical analysis is applied here to two of the real fugues from Bach’s Well-tempered Clavier (the ‘48’), that is, the fugues in which the subject is answered by an exact transposition by 7 pitch values (semitones) up, pitch class-equivalent to $12 - 7 = 5$ down. The interval class vectors indicate how many different pairs of pitch values are found at each distance apart, from 1 to 6 (7 and above being mapped to 5 and below). Of the major-key fugues in the 48, the majority have subjects using 6 of the 7 notes in the major scale. If the missing note value is 5 or 11 then the interval class vector is $(1,4,3,2,5,0)$, as found here for the C major fugue (the one that follows that prelude); Babbitt’s Theorem then guarantees maximum possible overlap between the notes of the subject and its answer: $v_5 = 5$. In the $F$♯ fugue from Book 2, some chromaticism allows an 8-pitch subject; again Bach’s (deliberate?) choice of pitches means that the answer has maximum overlap: $v_5 = 7$.

This simple but effective result is attributed by music theorists to the American composer and analyst Milton Babbitt who would presumably have first discovered and used it in the 1950s.

**Web link:** [www.mta.ca/faculty/arts-letters/music/pc-set_project/pc-set_new/](http://www.mta.ca/faculty/arts-letters/music/pc-set_project/pc-set_new/). The Bach extracts are typeset in Lime.

**Further reading:** *Music and Mathematics* by John Fauvel, Raymond Flood and Robin Wilson, Oxford University Press, 2003 (chapter 9).