Catalan’s Conjecture (Mihăilescu’s Theorem) Let \( x, y, p, q \) be positive integers satisfying \( x^p - y^q = 1 \). Then \( x = q = 3 \) and \( y = p = 2 \).

In other words, 8 and 9 are the only nontrivial instance of consecutive perfect powers. We may restrict attention to prime powers since a solution to, say, \( x^4 = y^{15} + 1 \), would give a prime power solution too: \( (x^2)^3 = (y^3)^5 + 1 \). Even if we ask for two perfect powers whose difference is equal to some specific \( t \neq 1 \), solutions appear to very scarce: those shown above-right are the only ones for \( t \leq 10 \) when \( x, y, p, q \in \{2, \ldots, 100\} \).

(The diophantine equation \( x^p - y^q = 6 \) has no solutions in this range, indeed, I do not know if any exist at all.) In fact, a conjecture of Subbayya Sivasankaranarayana Pillai from the 1930’s asserts that, for any positive integer \( t \), there are only finitely many values of \( x, y, p, q \geq 2 \) solving \( x^p - y^q = t \).

As with Fermat’s Last Theorem, the solution to this 1844 conjecture of Eugène Catalan, was assembled over a long period of time. Victor Lebesgue quickly established that \( q \neq 2 \) (1850) but it then took over a hundred years before Chao Ko, in about 1960, settled the other quadratic case: \( p \neq 2 \), except when \( x = 3 \). This left \( p, q \) odd primes, and, expressing the equation as \( (x - 1) \times (x^p - 1)/(x - 1) = y^q \), it could be shown that the greatest common divisor of the two left-hand factors was either 1 or \( p \).

The former case had just been eliminated by J.W.S Cassels in 1960; only ‘Case II’, \( \gcd = p \), remained. It was this last, formidable, hurdle that Mihăilescu surmounted. In 2000, he showed that \( p \) and \( q \) would have to be a so-called ‘Wieferich pair’: satisfying \( p^{q-1} \equiv 1 \pmod{q^2} \) and \( q^{p-1} \equiv 1 \pmod{p^2} \); then, in 2002, he showed that such solutions were an impossibility.


**Further reading:** *Catalan’s Conjecture* by René Schoof, Springer-Verlag, London, 2008.