THEOREM OF THE DAY

The Andrews–Garvan–Dyson Crank *Let n be a non-negative integer and π a partition of n. Denote by λ(π) the largest part of π; by M(π) the number of '1's in π; and by N(π) the number of parts of π which exceed the value of M(π). Define the* crank *of π by*

$$\operatorname{crank}(\pi) = \begin{cases} \lambda(\pi) & \text{if } M(\pi) = 0, \\ N(\pi) - M(\pi) & \text{otherwise.} \end{cases}$$

Now for any integer n congruent to 6 mod 11, the residue classes mod 11 of the crank *values of the partitions of n are equinumerous.*

Crank function residues mod 11 for the 297 partitions of 17.

5	7	8	9	9	10	10	0	0	0	0	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3
3	3	4	4	4	4	4	4	4	5	4	4	4	4	5	5	5	5	6	5	5	5	6	6			
5	6	6	6	6	7	6	6	7	6	7	7	6	7	6	7	6	7	6	7	7	8	7	7	8	7	8
7	8	8	8	8	7	8	8	9	8	7	8	8	9	8	8	8	9	9	8	8	9	8	9	9	10	9
9	9	10	10	9	8	9	10	9	0	10	10	10	9	9	10	9	10	10	9	0	0	10	10	10	10	9
10	10	0	10	1	0	0	10	1	1	0	0	0	10	2	1	1	1	0	1	0	0	0	10	3	2	2
1	2	1	1	1	0	1	1	0	1	0	0	0	10	7	6	6	5	5	5	4	5	4	4	4	3	5
4	4	3	4	3	3	3	2	4	4	3	3	3	2	3	3	2	2	2	2	1	4	3	3	3	2	3
2	2	2	1	3	2	2	2	1	2	1	1	1	1	0	3	5	4	7	3	6	5	9	5	4	8	7
6	0	4	7	6	5	10	5	9	8	7	2	3	6	5	9	4	8	7	6	1	7	6	0	5	10	9
8	4	5	4	8	7	6	0	6	5	10	9	8	7	3	5	9	8	7	2	7	6	1	0	10	9	6

If each partition of *n* is prefixed by a '1' it becomes a partition of *n* + 1. In addition there are 'new' partitions of *n* + 1 which have no '1's (so $M(\pi) = 0$ for each such partition π). The colour coding in the table on the left reflects this: beginning top-left, if $p(n)$ denotes the number of partitions of *n* then $p(1) = 1$; $p(2) = p(1) + 1$; $p(3) = p(2) + 1$; $p(4) = p(3) + 2$, and so on. To give a concrete example, the 11 partitions of 6 are obtained by: (a) prefixing with a 1 each of the seven partitions of 5: (1 1 1 1 1), (1 1 1 2), (1 1 3), (1 2 2), (1 4), (2 3), (5); and then (b) identifying the '*M* = 0' partitions of 6: (2 2 2), (2 4), (3 3) and (6). We can list the crank value of each partition so constructed: crank((1 1 1 1 1 1)) = 0 − 6 = −6 and, continuing, we get −4, −3, −2, −1, 1, 0, 2, 4, 3, and 6. And modulo 11, this gives: 5, 7, 8, 9, 10, 1, 0, 2, 4, 3, 6; each residue occurring the same number of times. Our table, however, goes all the way up to the 297 partitions of 17; by this time a further twelve '1's had been prefixed to the above partitions of 5. The fifth value in row 1 of the table, for example, is crank((1 1 1 1 1 1 1 1 1 1 1 1 1 4)) mod 11 = −13 mod 11 = 9. And 9 occurs a total of 27 times in the table, as does each other residue mod 11.

The crank function provides a direct proof of the Ramanujan partition congruence $p(11t + 6) \equiv 0 \pmod{11}$. Indeed, it also works for the others of his famous triptych: $p(5t + 4) \equiv 0 \pmod 5$ and $p(7t + 5) \equiv 0 \pmod 7$. Freeman Dyson proposed the crank's existence while an undergraduate at Cambridge in the early 1940s. It was more than 40 years later that George Andrews and Frank Garvan invented an actual crank function, as specified in today's theorem, and proved that it worked.