The Andrews–Garvan–Dyson Crank

Let \( n \) be a non-negative integer and \( \pi \) a partition of \( n \). Denote by \( \lambda(\pi) \) the largest part of \( \pi \); by \( M(\pi) \) the number of ‘1’s in \( \pi \); and by \( N(\pi) \) the number of parts of \( \pi \) which exceed the value of \( M(\pi) \). Define the crank of \( \pi \) by

\[
\text{crank}(\pi) = \begin{cases} 
\lambda(\pi) & \text{if } M(\pi) = 0, \\
N(\pi) - M(\pi) & \text{otherwise}.
\end{cases}
\]

Now for any integer \( n \) congruent to 6 mod 11, the residue classes mod 11 of the crank values of the partitions of \( n \) are equinumerous.

<table>
<thead>
<tr>
<th>Crank function residues mod 11 for the 297 partitions of 17.</th>
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<tr>
<td>5 7 8 9 9 10 10 0 0 0 0 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 33 3 4 4 4 4 4 4 4 5 4 4 4 4 5 5 5 5 6 5 5 5 6 5 5 6 6 7 7 7 7 7 7 7 7 7 7 7 7 7 8 8 8 8 8 8 8 8 8 8 8 8 8 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9</td>
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The crank function provides a direct proof of the Ramanujan partition congruence \( p(11t + 6) \equiv 0 \pmod{11} \). Indeed, it also works for the others of his famous triptych: \( p(5t + 4) \equiv 0 \pmod{5} \) and \( p(7t + 5) \equiv 0 \pmod{7} \). Freeman Dyson proposed the crank’s existence while an undergraduate at Cambridge in the early 1940s. It was more than 40 years later that George Andrews and Frank Garvan invented an actual crank function, as specified in today’s theorem, and proved that it worked.
