The Bungers–Lehmer Theorem on Cyclotomic Coefficients

The cyclotomic polynomials, taken over all products of three distinct primes, contain arbitrarily large coefficients.

The image on the right depicts the 105-th roots of unity, $\sqrt[105]{1}$, whose values are given by $e^{\tau ik/105} = \cos\frac{\tau k}{105} + i\sin\frac{\tau k}{105}$ $(k \geq 0, \tau = 2\pi)$. Those marked with a star are *primitive roots*: when k is coprime to 105, i.e. $\mathrm{GCD}(k, 105) = 1$, the successive powers of $e^{\tau ik/105}$ will generate every other 105th root. The n-th cyclotomic polynomial $\Phi_n(x)$ is the unique monic polynomial whose roots are precisely the primitive n-th roots of unity. Now $105 = 3 \times 5 \times 7$ and there are 48 integers less than 105 and coprime to 105 (the sequence $1, 2, 4, 8, 11, \ldots, 104$, anticlockwise from far right on the horizontal axis). We denote this by $\varphi(105) = 48$, φ being the Euler totient function. Now we have $\Phi_{105}(x) = (x - e^{\tau i/105})(x - e^{2\tau i/105})(x - e^{4\tau i/105})\cdots(x - e^{104\tau i/105})$; remarkably the cyclotomic polynomials, defined in this way, always expand to give a polynomial of degree $\varphi(n)$ all of whose coefficients are integers! Indeed, for our example we get:

$$\begin{aligned}\Phi_{105}(x) = {} & 1 + x + x^2 - x^5 - x^6 - 2x^7 - x^8 - x^9 + x^{12} + x^{13} + x^{14}\\ & + x^{15} + x^{16} + x^{17} - x^{20} - x^{22} - x^{24} - x^{26} - x^{28}\\ & + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} - x^{39} - x^{40}\\ & - 2x^{41} - x^{42} - x^{43} + x^{46} + x^{47} + x^{48}.\end{aligned}$$

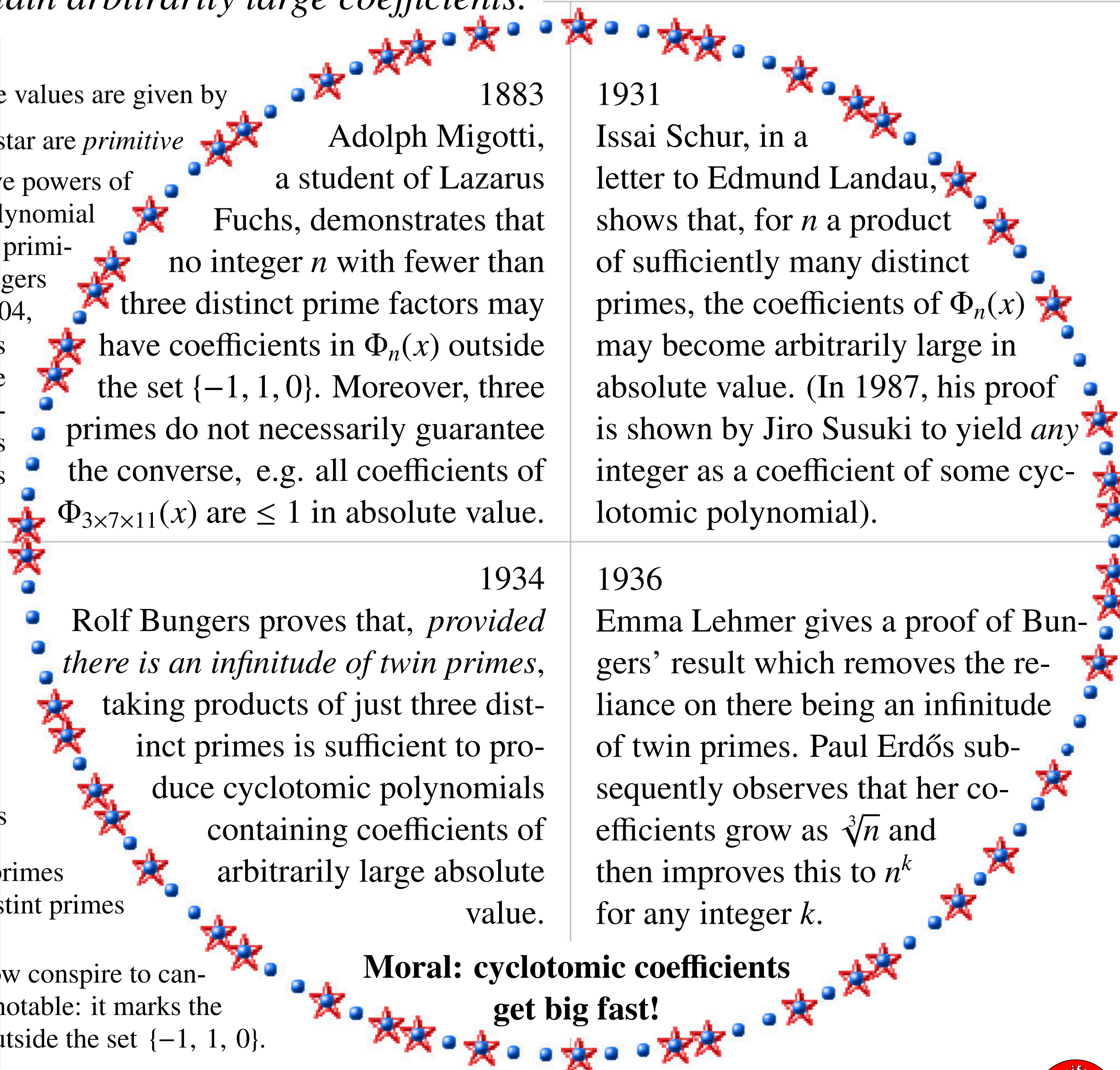
An equivalent definition of $\Phi_n(x)$ sheds light on this phenomenon:

$$\Phi_n(x) = \prod_{d|n}(1 - x^{n/d})^{\mu(d)},$$

the product running over all divisors d of n, with μ being the Möbius function:

$$\mu(k) = \begin{cases} -1 & k \text{ is a product of an odd number of distint primes}\\ 1 & k \text{ is 1 or a product of an even number of distint primes}\\ 0 & k \text{ is not square-free}\end{cases}$$

(giving factors in the product of negative index, which must somehow conspire to cancel). The coefficient of -2 appearing in the expansion of $\Phi_{105}(x)$ is notable: it marks the first occurrence of a cyclotomic polynomial coefficient which lies outside the set $\{-1, 1, 0\}$.



1883
Adolph Migotti, a student of Lazarus Fuchs, demonstrates that no integer n with fewer than three distinct prime factors may have coefficients in $\Phi_n(x)$ outside the set $\{-1, 1, 0\}$. Moreover, three primes do not necessarily guarantee the converse, e.g. all coefficients of $\Phi_{3\times7\times11}(x)$ are ≤ 1 in absolute value.

1931
Issai Schur, in a letter to Edmund Landau, shows that, for n a product of sufficiently many distinct primes, the coefficients of $\Phi_n(x)$ may become arbitrarily large in absolute value. (In 1987, his proof is shown by Jiro Susuki to yield *any* integer as a coefficient of some cyclotomic polynomial).

1934
Rolf Bungers proves that, *provided there is an infinitude of twin primes*, taking products of just three distinct primes is sufficient to produce cyclotomic polynomials containing coefficients of arbitrarily large absolute value.

1936
Emma Lehmer gives a proof of Bungers' result which removes the reliance on there being an infinitude of twin primes. Paul Erdős subsequently observes that her coefficients grow as $\sqrt[3]{n}$ and then improves this to n^k for any integer k.

Moral: cyclotomic coefficients get big fast!

Web link: www.maths.lancs.ac.uk/~jameson/cyp.pdf. Lehmer's original paper is available at www.projecteuclid.org/euclid.bams/1183498920.

Further reading: *Polynomials* by Victor Prasolov, Springer, 2nd printing, 2009, chapter 3.