Hyperperfect numbers

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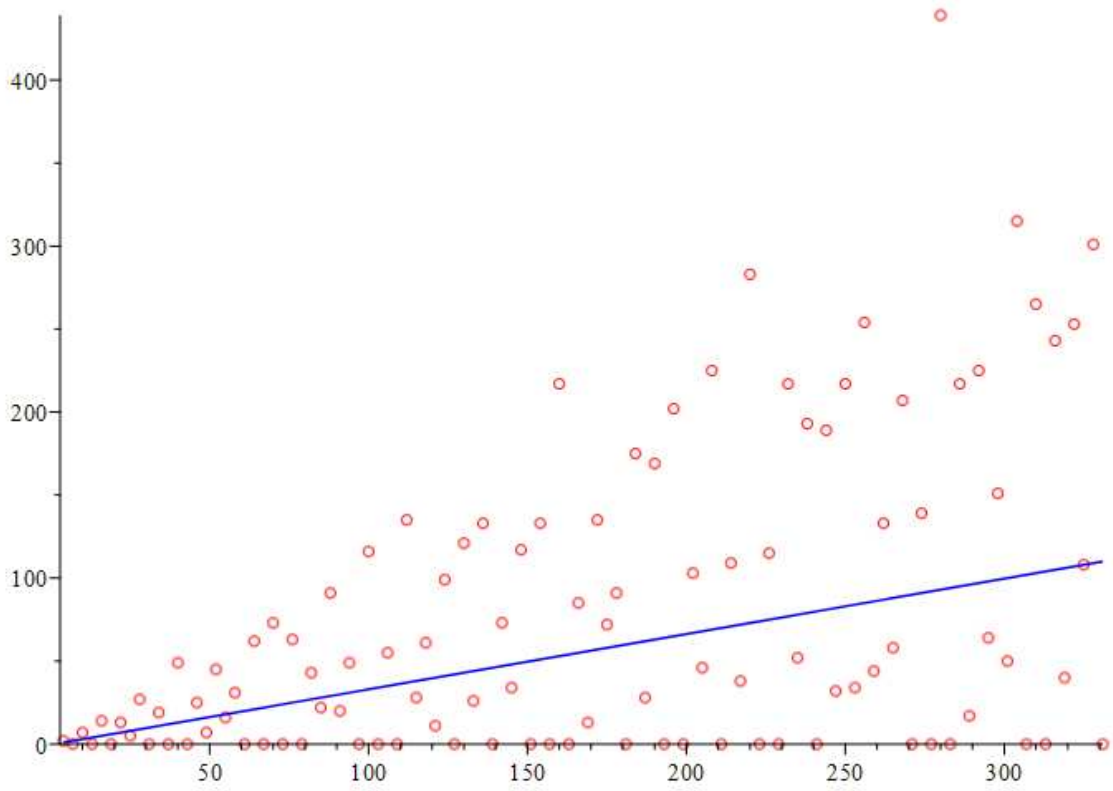


Mersenne Numbers

This talk is mostly about the expression

$$(1+k)^{m-1}\left((1+k)^m - k\right). \tag{1}$$

If $k = 1$ the second factor is a Mersenne number and is known (as of July 2026) to be prime for 52 values of m.

For all these values, expression (1) evaluates to a perfect number, that is, one which is equal to exactly half the sum of its divisors.

For $m = 2$, for example, the value is 6 which is half of $1+2+3+6$. Here are some old friends, the first few perfect numbers:

$$6, 28, 496, 8128, 33550336, 8589869056, 137438691328, \ldots.$$

Hyperperfect Numbers: 1st definition

For a positive integer n, the sum of divisors function, $\sigma(n)$, is defined to be the sum of the positive divisors of n. A perfect number is thus a value of n for which $\sigma(n) = 2n$.

Then, for the equation

$$\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}, \tag{2}$$

the solutions when $k = 1$ are the perfect numbers.

The solutions for a general positive integer value of k are called the k-hyperperfect numbers. They were introduced in Daniel Minoli and Robert Bear, "Hyperperfect numbers", *Pi Mu Epsilon J.*, Vol. 6, No. 3, 1975, pp. 153-157.

Hyperperfect Numbers: 2nd definition

Equation (2) rearranges to give the alternative, more often used, definition: n is a k-hyperperfect number if and only if it satisfies the equation:

$$n = 1 + k\left(\sigma(n) - n - 1\right). \tag{3}$$

Calculations using equations (2) and (3) are often made easy by the basic properties of the σ function:

Lemma 1 For n a positive integer

1. if n is prime then $\sigma(n) = 1 + n$;
2. $\sigma(n)$ is multiplicative: if n is a product of two coprime numbers, say $n = ab$, then $\sigma(n) = \sigma(a).\sigma(b)$.

Hyperperfect Numbers: 3rd definition

A third way of defining the hyperperfect numbers is rather more natural but uses a version of the σ function that is less standard (no longer multiplicative, for example): let $\sigma^*(n)$ denote the sum of the proper, nontrivial divisors, that is, $\sigma^*(n) = \sigma(n) - 1 - n$.

The perfect numbers are then those n for which $n = 1 + \sigma^*(n)$, and n is a k-hyperperfect number if and only if

$$n = 1 + k\sigma^*(n). \tag{4}$$

We use this definition to prove a simple but striking result of Minoli and Bear:

Lemma 2 If n has a divisor strictly between 1 and $1 + k$ then n cannot be k-hyperperfect.

Hyperperfect Numbers: a Lemma

Lemma 2 If n has a divisor strictly between 1 and $1+k$ then n cannot be k-hyperperfect.

Proof. Suppose that d divides n with $1 < d \leq k$. We observe that d cannot equal n, otherwise $n \leq k$ and it is impossible to have $n = 1 + k\sigma^*(n)$.

Suppose that d is a proper divisor of n. Then n/d is also a proper divisor of n. But then the definition of k-hyperperfect gives

$$n = 1 + k \sum_{d|n, 1<d<n} d = 1 + k\left(\frac{n}{d} + \ldots\right) > n,$$

since $d \leq k$. So n cannot be k-hyperperfect. □

The Euclid–Euler Theorem

Lemma 2 If n has a divisor strictly between 1 and $1 + k$ then n cannot be k-hyperperfect.

Lemma 2 is striking because it follows that no k-hyperfect number can be even when $k > 1$; whereas it is famously an open question whether there exist any 1-hyperperfect (that is, perfect) numbers that are *not* even. In other words, it is an open question whether or not the following celebrated injective mapping from Mersenne primes to perfect numbers extends to a surjection:

Theorem 3 (Euclid–Euler): An even positive integer n is perfect if and only if it can be written as

$$n = 2^{m-1}\left(2^m - 1\right), \tag{5}$$

for some m for which $2^m - 1$ is prime.

From Euclid–Euler to Hyperperfect

The great appeal of Minoli and Bear's definition lies in the following, as they put it, "astonishing relation" to the Euclid–Euler theorem:

Theorem 4 (Minoli and Bear, Nash): For $k \geq 1$, let $n = (1+k)^{m-1}((1+k)^m - k)$, where $(1+k)^m - k$ is prime. Then n is a k-hyperperfect number if and only if $1+k$ is prime.

Minoli and Bear proved this result only for $k = 2$ but the proof for the general case follows in an almost identical fashion. It was written down in John C. M. Nash, "Hyperperfect numbers", *Pi Mu Epsilon J.*,Vol. 11, No. 5, 2001, p. 251 (by the way, this is Nash the son of John Forbes Nash).

Neither state the 'only if' part but this follows at once from Lemma 2 since if $1+k$ is not prime then n in the theorem has a divisor strictly less than $1+k$.

Beyond the Mersenne Numbers

The hypothesis of Theorem 4 for $k = 1$ matches the conclusion of the Euclid-Euler theorem and, as far as we know, accounts for all 1-hyperperfect numbers.

We might hope that this hypothesis would account for all k-hyperperfect numbers in every case where k is one less than a prime. This almost immediately fails because $7^2.383.3203$ is 6-hyperperfect.

In fact, for even values of k, things appear quite badly-behaved, the worse excesses being reported in Mariano Garcia, "Hyperperfect numbers with five and six different prime factors", *Fibonacci Quarterly*, Vol. 42, No. 4, 2004, pp. 292–294.

Hot Off The Press

The case $k = 2$ appeared promising: for fifty years the only known 2-hyperperfect numbers have been those specified in theorem 4.

However, in June 2026, on the SeqFAN Google Group ("a mailing list for Integer SEQuence FANs"), user Alex Violette announced the discovery of a 2-hyperperfect number that did not involve the Mersenne numbers):

$$3^{16} \cdot 129140227 \cdot 256572153302107 \cdot 22520539685553710729893866373.$$

The three factors following the 3^{16} are primes. This makes it easy to confirm, via the properties of the σ function, that Alex Violette's find satisfies the definition of 2-hyperperfect numbers.

k-Hyperperfect for odd k

The case of $k > 1$ odd is of course excluded from theorem 4. In this case we have a construction from Judson S. McCranie, "A study of hyperperfect numbers", *Journal of Integer Sequences*, Vol. 3, 2000, Article 00.1.3.

Theorem 5 (McCranie): For odd $k > 1$, let $p = (3k + 1)/2$ and $q = 3k + 4$ be prime. Then p^2q is k-hyperperfect.

McCranie conjectured that no other k-hyperperfect numbers exist for odd $k > 1$ and this conjecture still stands as of July 2026. Moreover, there are many odd k for which the hypothesis of theorem 5 cannot hold ($k = 5$, for example). So, conjecturally, there are values of k for which no k-hyperperfect numbers exist. In particular, no 5-hyperperfect numbers are known.

Experiments with odd k

We can explore particular values of k numerically.

Recall n is a k-hyperperfect number if and only if

$$n = 1 + k\sigma^*(n).$$

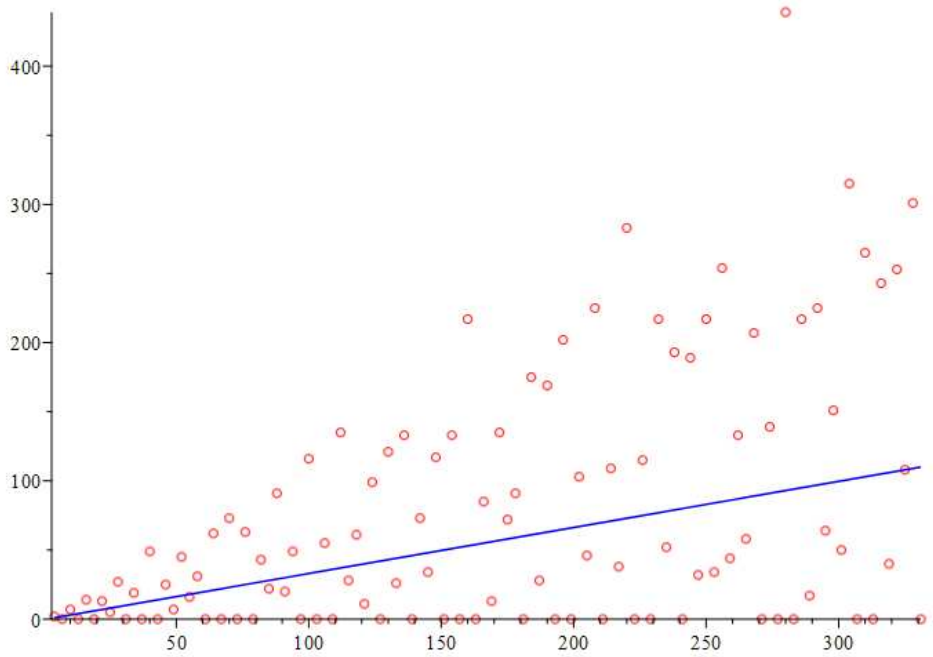
Thus, for n to be k-hyperperfect, we have $n = 1 + kt$ for some $t \geq 1$.

By the same definition, $1 + kt = 1 + k\sigma^*(kt + 1)$, so

$$\sigma^*(kt + 1) = t. \tag{6}$$

Experimenting with $k = 3$

The following plot explores the value $k = 3$. The straight line is $y = (x - 1)/3$. The value of $\sigma^*(3n + 1), n = 0, \ldots, 110$, is plotted against this line as a series of points. There is one value where the two plots coincide, indicating a 3-hyperperfect number. This is the point $(3t + 1, t)$ for $t = 108$, verifying identity (6) and confirming the assertion of theorem 5 that, with $p = (3k + 1)/2 = 5$ and $q = 3k + 4 = 13$ both prime, the value $p^2q = 325$ is a 3-hyperperfect number.



Experimenting with $k = 5$

The following plot shows data for $k = 5$. Despite some 'near misses' no points on the straight line $y = (x - 1)/5$ are σ^* values.

There are some seemingly linear distributions of σ^* values which invite investigation: perhaps these correspond to some variant of hyperperfect numbers whose definition is much easier to satisfy!