Euler’s Continued Fraction Correspondence

Let \((a_i)_{i \geq 0}\) be an infinite sequence of nonzero real or complex numbers. Let \(f_n\) denote the \(n\)-th partial sum of the sequence: \(f_n = \sum_{i=0}^{n} a_i\). Then \(f_n\) is also the \(n\)-th convergent of the continued fraction described below:

\[ a_0 + \frac{a_1}{1 + \frac{-a_2/a_1}{1 + \frac{-a_3/a_2}{1 + \frac{-a_4/a_3}{1 + \ldots}}}}. \]

Relative error \(|(f_n - \tau)/\tau|\), \(\tau = 2\pi\) in the \(n\)-th convergent \(f_n\) of the continued fraction described below, compared for \(n = 1, \ldots, 7\) by plotting the values as shares of a pie.

If we apply Euler’s correspondence to Nilakantha’s series with \(a_i = (-1)^{k-1}/k(2k+1)(2k+2)\) then we get

\[ \tau - \frac{6}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(2k+1)(2k+2)} = 1 + \frac{1.3.4}{12} + \frac{2.5.6}{12} + \frac{3.7.8}{12} + \ldots = 1 + \frac{2^2}{12} + \frac{6^2}{12} + \frac{10^2}{12} + \ldots, \]

giving \(\tau = 6 + \frac{4}{12} + \frac{1.3.4}{12} + \frac{2.5.6}{12} + \frac{3.7.8}{12} + \ldots = 6 + \frac{2^2}{12} + \frac{6^2}{12} + \frac{10^2}{12} + \ldots\), whose convergents are explored in the above pie chart.

Leonhard Euler discovered this correspondence in 1748. The above application to \(\tau\) (in a \(\pi\) version) was given by Douglas Bowman as an alternative derivation of a continued fraction published by Jerome Lange in 1999.

Web link: people.math.binghamton.edu/dikran/478/Ch7.pdf.

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