THEOREM OF THE DAY

Euler's Product Formula for $\zeta(s)$ *For any complex number s having real part* $Re(s) > 1$,

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots = \left(1 - \frac{1}{p_1^s}\right)^{-1}\left(1 - \frac{1}{p_2^s}\right)^{-1}\left(1 - \frac{1}{p_3^s}\right)^{-1}\cdots$$

where $p_1, p_2, p_3, \ldots$ *are the prime numbers.*

$$\text{Starting point: } \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots \qquad (1)$$

Take any prime p

$$\frac{1}{p^s} \times (1) : \zeta(s)\frac{1}{p^s} = \frac{1}{p^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots\right) \qquad (2)$$

$$\text{giving: } \zeta(s)\frac{1}{p^s} = \frac{1}{p^s} + \frac{1}{(2p)^s} + \frac{1}{(3p)^s} + \frac{1}{(4p)^s} + \ldots \qquad (3)$$

$$(1) - (3) : \zeta(s)\left(1 - \frac{1}{p^s}\right) = 1 + \ldots \text{no } \frac{1}{(kp)^s} \text{ terms!} \qquad (4)$$

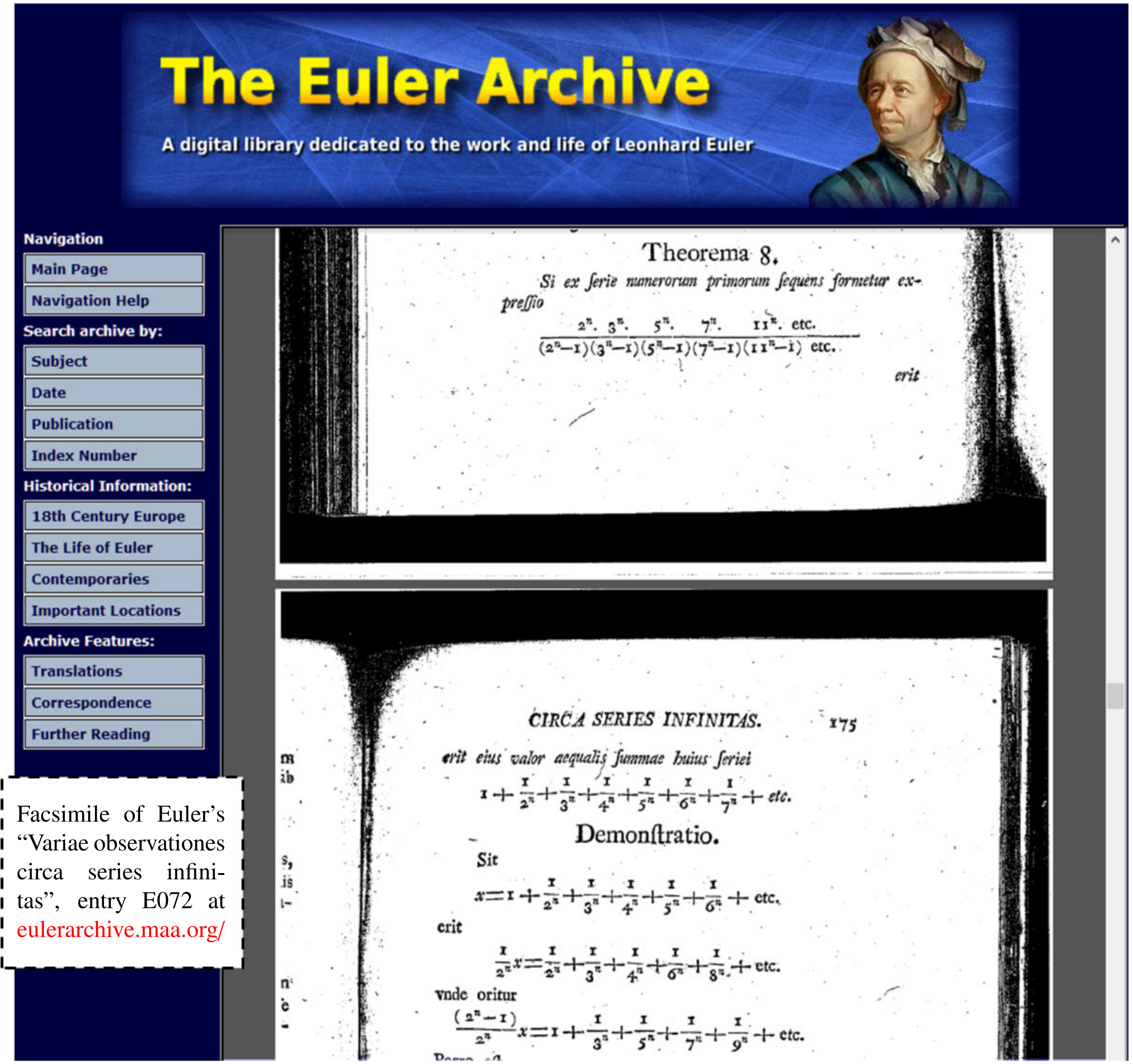
Repeat the above process for every prime p. On the left-hand side we get:

$$\zeta(s)\left(1 - \frac{1}{p_1^s}\right)\left(1 - \frac{1}{p_2^s}\right)\left(1 - \frac{1}{p_3^s}\right)\cdots = \zeta(s)\prod_p\left(1 - \frac{1}{p^s}\right)$$

All that will remain on the right is ... 1. Divide out:

$$\zeta(s) = \frac{1}{\prod_p\left(1 - \frac{1}{p^s}\right)} = \prod_p\left(1 - \frac{1}{p^s}\right)^{-1}.$$

With his formula, Euler initiates an extended analysis which culminates in a proof that the sum of the reciprocals of the primes diverges. It was presented to the St. Petersburg Academy in 1737 and was published by them in 1744 (see the facsimile on the right, courtesy of the wonderful Euler Archive, and spot Euler's misprint!) The convergence properties of the formula were not studied until the next century and the extension to complex numbers is due to Riemann in his famous 1859 "Über die Anzahl der Primzahlen".



Facsimile of Euler's "Variae observationes circa series infinitas", entry E072 at eulerarchive.maa.org/

Web link: www.maths.tcd.ie/pub/Maths/Courseware/428/ (notes by Timothy Murphy, Euler's formula is discussed in Primes-II).
Further reading: *How Euler Did It* by C. Edward Sandifer, MAA, 2007, Chapter 33.