The Polygonal Number Theorem

Robin Whitty

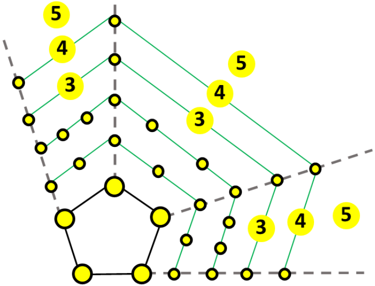
theoremoftheday.org

6 May, 2021

Polygonal Numbers

For $m \geq 1$, the k-th polygonal number of order $m+2$ is defined to be

$$P_m(k) = \frac{m}{2}\left(k^2 - k\right) + k, k \geq 0.$$



From $m+1$ vertices of an $(m+2)$-gon, add new vertices along 'rays', interpolating $1, 2, 3, \ldots,$ vertices.
For $m = 3$, pentagonal numbers: $1, 5, 12, 22, \ldots$.

Triangular Numbers

For $m = 1$, the polygonal numbers of order 3 are the triangular numbers

$$P_1(k) = \frac{1}{2}\left(k^2 + k\right), k \geq 0.$$

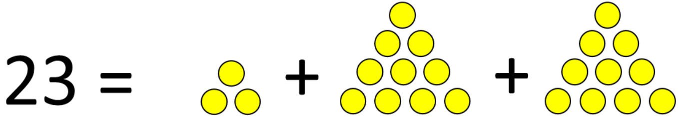
1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

...

The triangular numbers appear on the 2nd diagonal of Pascal's triangle. (The sum of their reciprocals converges, as is the case with all diagonals except the 0th and 1st - Nick Hobson, M500, Issue 216.)

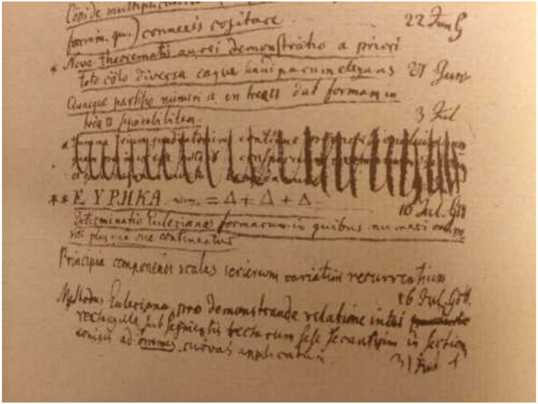
Fermat's Polygonal Number Conjecture

Fermat, in a letter to Mersenne in 1636, asserted that, for all positive integer m, every nonnegative integer is a sum of $m+2$ polygonal numbers of order $m+2$



Fermat $m = 1$: Gauss's Eureka Theorem

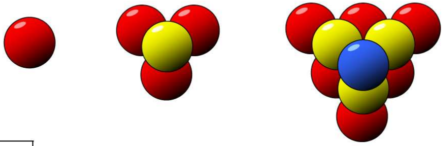
Gauss proved the case $m = 1$ of Fermat's conjecture on July 10, 1796, as per the following entry in his diary:



Equivalently, if $n \equiv 3 \mod 8$ then n can be written as $n = x^2 + y^2 + z^2$ for odd integers x, y, z.

Tetrahedral Numbers

The third diagonal of Pascal's triangle are the **tetrahedral numbers**, being the number of spheres that can be densely packed in a triangular pyramid.

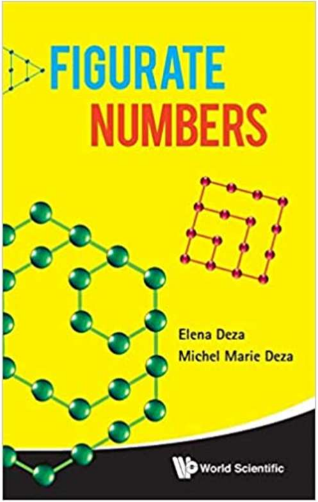


1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

. . .

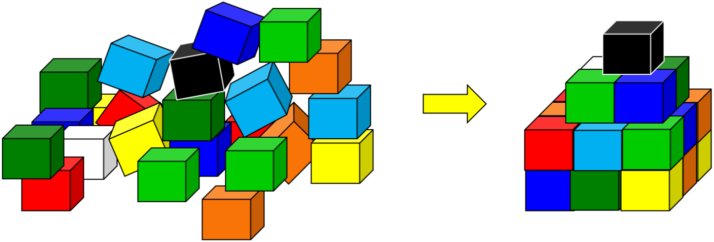
The diagonals generally generalise the triangular numbers to **figurate** numbers: tetrahedral, 4-simplex, etc, although definitions vary.

Figurate Numbers



Fermat $m = 2$: Square Numbers

When $m = 2$ we have the polygonal numbers of order 4, the **square numbers**. Any positive integer may be written as a sum of at most four square numbers. This was known to Diophantus of Alexandria and was first explicitly asserted by Claude Gaspard Bachet de Méziriac,who translated Diophantus's Arithmetica into Latin in 1621. The first proof is due to Lagrange in 1770.

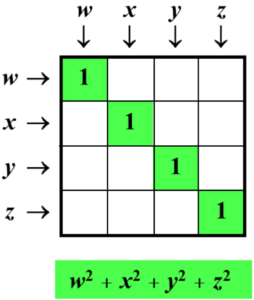


Any collection of blocks may be arranged as a square pyramid of height at most 4 blocks. E.g. $23 = 3^2 + 3^2 + 2^2 + 1^2$.

The Fifteen Theorem

In 1993, John Conway and his student William Schneeberger proved:

> *If a positive-definite quadratic form defined by a symmetric, integral matrix takes each of the values 1, 2, 3, 5, 6, 7, 10, 14, 15, then it takes all positive integer values.*



w	*x*	*y*	*z*	
0	0	0	1	**1**
0	0	1	1	**2**
0	1	1	1	**3**
0	0	1	2	**5**
0	1	1	2	**6**
1	1	1	2	**7**
1	1	2	2	**10**
0	1	2	3	**14**
1	1	2	3	**15**

Waring's Problem

Coincidentally with Lagrange's proof of the four-squares theorem, Edward Waring proposed (and, in 1909, Hilbert confirmed, nonconstructively):

> *For each positive integer k, there is a positive integer $g(k)$ such that every nonnegative integer may be written as a sum of at most $g(k)$ integers raised to power k.*

E.g. $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$: one of only two cases where $g(3) = 9$ nontrivial cubes are required. It is thought that six are sufficient for large n ($G(3) = 6$).

Theorem (1940s): If $\left\{\left(\frac{3}{2}\right)^n\right\} \leq 1 - \left(\frac{3}{4}\right)^n$, where $\{\,.\,\}$ denotes the fractional part of a real number, then

$$g(n) = 2^n + \left[\left(\frac{3}{2}\right)^n\right] - 2,$$

where $[\,.\,]$ denotes the integer part of a real number.

Fermat $m \geq 3$: Cauchy's Lemma

Cauchy (1815) showed that Gauss's Eureka Theorem implies:

If a and b are odd positive integers satisfying

$$b^2 - 4a < 0 \quad \textit{and} \quad 0 < b^2 + 2b - 3a + 4,$$

then there exist nonnegative integers s, t, u, v such that

$$a = s^2 + t^2 + u^2 + v^2 \quad \textit{and} \quad b = s + t + u + v.$$

E.g. $a = 259$, $b = 29$,

$$b^2 - 4a = 841 - 1036 < 0$$
$$b^2 + 2b - 3a + 4 = 841 + 58 - 777 + 4 > 0$$
$$259 = 13^2 + 7^2 + 5^2 + 4^2$$
$$29 = 13 + 7 + 5 + 4$$

Fermat $m \geq 3$: Cauchy's Theorem

Cauchy (1815): If $m \geq 3$ then every nonnegative number can be written as a sum of at most $m+2$ polygonal numbers of order $m+2$.

E.g.
$375 = P_3(13)+P_3(7)+P_3(5)+P_3(4)+P_3(1) = 247+70+35+22+1$.
(Recall

$$P_m(k) = \frac{m}{2}\left(k^2 - k\right) + k, k \geq 0.)$$

Subsequent work, notably by Jean François Théophile Pépin, constructed explicit order $m+2$ polygonal representations for all integers $n < 120m$.

Fermat $m \geq 3$: Melvyn Nathanson's proof

1. Assume that $m \geq 3$. Choose an odd positive integer b such that
   1.1 We can write $n \equiv b + r \mod m, 0 \leq r \leq m - 2$; and
   1.2 If $a = 2\left(\dfrac{n-b-r}{m}\right) + b$, an odd positive integer by virtue of (1.1), then

$$b^2 - 4a < 0 \quad \text{and} \quad 0 < b^2 + 2b - 3a + 4. \qquad (*)$$

2. Invoke **Cauchy's Lemma:** *If a and b are odd positive integers satisfying* $(*)$ *then there exist nonnegative integers* s, t, u, v *such that*

$$a = s^2 + t^2 + u^2 + v^2 \quad \text{and} \quad b = s + t + u + v.$$

3. From the definition of a in (1.2), write $n = \dfrac{m}{2}(a - b) + b + r$ $= \frac{m}{2}(s^2 - s) + s + \ldots + \frac{m}{2}(v^2 - v) + v + r.$

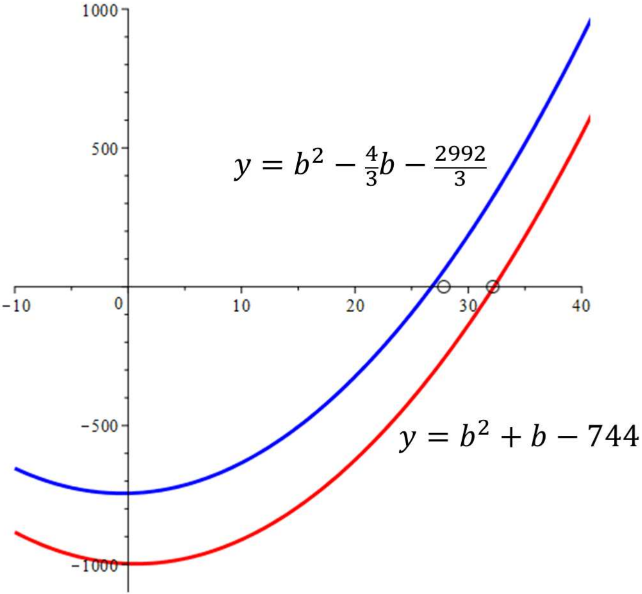
Fermat $m \geq 3$: Nathanson's proof, the small print

1. Assume that $m \geq 3$. Choose an odd positive integer b such that
   1.1 We can write $n \equiv b + r \mod m, 0 \leq r \leq m - 2$; and
   1.2 If $a = 2\left(\frac{n-b-r}{m}\right) + b$, an odd positive integer by virtue of (1.1), then

$$b^2 - 4a < 0 \quad \text{and} \quad 0 < b^2 + 2b - 3a + 4. \qquad (*)$$

For a given n and m an interval for b may be expressed, via the quadratic formula, purely in terms of n and m. Namely, the interval $[1/2 + \sqrt{6(n/m) - 3}, 2/3 + \sqrt{8(n/m) - 8}]$ is guaranteed to be bounded by the zeros of the quadratics and to have length at least 4 for $n \geq 120m$.
Any interval of length 4 must contain two odd integers which together contain a complete set of residues mod m.

Nathanson's Interval



Legendre's Theorem (1832)

As n gets larger, the gap between the roots of the two quadratics in Nathanson's proof, as illustrated in the preceding slide, gets larger. This means the congruence $b \equiv b + r$ can be achieved with smaller and smaller values of r. Eventually $r = 0$ is always possible, for odd m, while for even m the least possible r oscillates between 0 and 1. Since r contributes all the polygonal numbers in Nathanson's proof except the first four, this gives an improvement on Cauchy's 1815 theorem, due to Legendre:

Let $m > 3$. If m is odd, then every sufficiently large integer is the sum of four polygonal numbers of order $m + 2$. If m is even, then every sufficiently large integer is the sum of five polygonal numbers of order $m + 2$, one of which is either 0 or 1.

Some links

- Nick Hobson, "Solution 213.1 - Pascal triangle sums", M500, Issue 216, pp. 1–2, m500.org.uk/magazine/.
- Lagrange's Four-Squares Theorem at theoremoftheday.org: www.theoremoftheday.org/Theorems.html#11
- Elena Deza and Michel Marie Deza, *Figurate Numbers*, World Scientific, 2012. Details and online reviews at www.theoremoftheday.org/Resources/Bibliography.htm#ElenaDeza
- The Fifteen Theorem at theoremoftheday.org: www.theoremoftheday.org/Theorems.html#79
- The Polygonal Number Theorem and Nathanson's proof are described here at theoremoftheday.org: www.theoremoftheday.org/Theorems.html#262