THEOREM OF THE DAY

The Polygonal Number Theorem For any integer \( m > 1 \), every non-negative integer \( n \) is a sum of \( m + 2 \) polygonal numbers of order \( m + 2 \).

For a positive integer \( m \), the polygonal numbers of order \( m + 2 \) are the values

\[
P_m(k) = \frac{m}{2} (k^2 - k) + k, \quad k \geq 0.
\]

The first case, \( m = 1 \), gives the triangular numbers, 0, 1, 3, 6, 10, ... A general diagrammatic construction is illustrated on the right for the case \( m = 3 \), the pentagonal numbers: a regular \((m+2)\)-gon is extended by adding vertices along ‘rays’ of new vertices from \((m+1)\) vertices with 1, 2, 3, ... additional vertices inserted between each ray.

How can we find a representation of a given \( n \) in terms of polygonal numbers of a given order \( m + 2 \)? How do we discover, say, that \( n = 375 \) is the sum \( 247 + 70 + 35 + 22 + 1 \) of five pentagonal numbers? What follows a piece of pure sorcery from the celebrated number theorist Melvyn B. Nathanson!

1. Assume that \( m \geq 3 \). Choose an odd positive integer \( b \) such that

   (1) We can write \( n \equiv b + r \ (\text{mod } m) \), \( 0 \leq r \leq m - 2 \); and

   (2) If \( a = 2 \left( \frac{n - b - r}{m} \right) + b \), an odd positive integer by virtue of (1), then

   \[
b^2 - 4a < 0 \quad \text{and} \quad 0 < b^2 + 2b - 3a + 4.
\]

2. Invoke Cauchy’s Lemma: If \( a \) and \( b \) are odd positive integers satisfying (*) then there exist nonnegative integers \( s, t, u, v \) such that

   \[
a = s^2 + t^2 + u^2 + v^2 \quad \text{and} \quad b = s + t + u + v.
\]

3. From the definition of \( a \) in step 1(2), write

   \[
n = \frac{m}{2} (a - b) + b + r
   \]

   \[
   = \frac{m}{2} (s^2 - s + s + \ldots + \frac{m}{2} (v^2 - v) + v + r.
\]

   \[
   = \frac{m}{2} (247 + 70 + 35 + 22 + 1)
\]

   \[
   = 375
\]

   \[
   = 29
\]

   \[
   = 29 \equiv 29 + 1 \ (\text{mod } 3)
\]

   \[
   = 259
\]

   \[
   = 841 - 1036 < 0, \quad 0 < 841 + 58 - 777 + 4
\]

   \[
   = 259 = 13^2 + 7^2 + 5^2 + 4^2 \quad \text{and} \quad 29 = 13 + 7 + 5 + 4
\]

   \[
   = 375 = 247 + 70 + 35 + 22 + 1
\]

   \[
   = 375
\]

   \[
   = 11-12/Mtl-To-numbertheory/ (11.45 on Sunday October 9)
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Created by Robin Whitty for www.theoremoftheday.org