THEOREM OF THE DAY

The Friedlander–Iwaniec Theorem There are infinitely many prime numbers of the form \( m^2 + n^4 \), for positive integers \( m \) and \( n \).

Number theorists are morally certain that any reasonable polynomial \( f(x_1, \ldots, x_t) \), in several positive integer variables and with integer coefficients, will take infinitely many prime values. Of course \( f \) must not factorise, and there are obvious so-called ‘local conditions’, e.g. \( f(x) = x(x + 1) + 2 \) is excluded because one of \( x \) and \( x + 1 \) is even, forcing \( f(x) \) to be even. To start with the one-variable linear prototype: \( f(x) = ax + b \) produces infinitely many primes if and only if \( a \) and \( b \) are coprime. Legendre asserted this in 1785; its proof sixty years later by Dirichlet marks the birth of analytic number theory. Taking \( a = 4 \) and \( b = 1 \), we see that infinitely many primes have the form \( 4k + 1 \) and these, as asserted by Girard and Fermat in the 16th century, are precisely the prime values of \( f(x_1, x_2) = x_1^2 + x_2^2 \). Although not easy to prove, Dirichlet’s result is easy to achieve in the sense that the sequence \( ax + b, x = 1, \ldots, N \), accounts for a proportion of about \( 1/a \) of the set \( \{1, \ldots, N\} \). Up to a constant multiple this means that \( N^1 \) values from \( \{1, \ldots, N\} \) are produced: we have marked this with the dashed line \( y = N \) on the chart above left. The polynomial \( x_1^2 + x_2^2 \) is less generous, but the proportion, determined by Landau and Ramanujan, is nearly linear: this is marked as the dashed line \( y = KN/\sqrt{\ln N} \), closely shadowing the actual count of integers \( \leq N \) representable as \( m^2 + n^4 \), displayed on our chart up to \( N = 10^4 \). Contrast this with the other three polynomials: the sequences of integer values they produce constitute only a fraction of \( N^\alpha \) of \( \{1, \ldots, N\} \), with \( \alpha < 1 \) in each case. Such sequences are known as ‘thin’.

Friedlander and Iwaniec used sophisticated prime ‘sieving’ methods to give the first proof that a thin polynomial sequence could contain infinitely many primes, inspiring Heath-Brown’s proof that there are infinitely many primes which are sums of three cubes.

Read Chapter 1 at press.princeton.edu/titles/8858.html.